CHAPTER 1 – The Measurement of Interest

1.1 ACCUMULATION FUNCTIONS AND THE TIME VALUE OF MONEY

Time Value of Money

Everything we will learn in this course is based upon the concept of the time value of money. **Time value of money** refers to the idea that receiving a certain amount of money today is worth more than receiving the same amount of money at some future time. To illustrate the intuition behind this idea, consider the following scenario.

> Assume that you win a small lottery. You are given the choice of two payments: $100 right now, or $$K$ to be paid one year from now. Let's consider under what conditions you would be enticed to take the later payment.
>
> If $K < 100$, then you would certainly take the earlier payment. It would not make sense to wait one year to receive less money. Even if $K = 100$, you should almost surely take the immediate payment. The payments are in the same amount, so there is no incentive for you to delay receipt of the payment. The *amounts* of the immediate payment and the delayed payment are the same, but you would likely attribute some additional worth to being able to receive the money immediately. Thus, we can say that the *time value* of a payment of $100 to be received immediately is greater than the *time value* of the payment of $100 delivered one year from now.
>
> It is clear that for you to consider taking the later payment, it would need to be in an amount larger than $100. Take a moment to try to consider how large K would need to be for you to choose to take the later payment rather than receiving $100 immediately.
>
> For the sake of discussion, let's assume that you have decided that you would take the earlier payment if $K < 120$, and you would take the later payment if $K > 120$. Let's also assume that if $K = 120$, then you would be equally happy with either payment. If this is the case, then a payment of $100 today would have the same inherent worth to you as a payment of $120 paid one year from now. We could say that the two payments have the same *time value* to you, even if they are in different amounts.
>
> It is entirely possible that someone else might have a different perspective on the relative value of these payments. Someone who has little immediate need for additional money might be more willing to take a later payment, and would thus require a smaller incentive to delay the payment. Such an individual might consider $100 paid today to have the same time value as $105 paid one year from now. This person would gladly trade $100 today in order to receive $110 one year from now, whereas you would refuse such an offer.

Although the exact way in which time effects the perceived value of a payment is subjective and can vary from one person to the next, it is hopefully clear that the time at which a payment occurs does effect the inherent value of that payment.

When two entities enter into an agreement that involves an exchange of money at different times, they will generally need to decide ahead of time on a method for determining the time value of money so that they can compare the value of payments occurring at different times. This goal can be achieved through the use of accumulation and amount functions, which we will introduce soon.

Interest

The most common application of the time value of money is interest. Let's assume that an amount P is borrowed, under the condition that the borrower will repay the loan at some point in the future. The concept of the time value of money dictates that for this arrangement to be fair to the lender, the amount repaid must be greater than the initial amount borrowed. Let's denote the amount repaid as $P+I$. The quantity P is the original loan amount, which is also referred to as the **principal** of the loan. The amount I is the **interest** on the loan, and can be thought of as a fee that the borrower pays to the lender to compensate the lender for temporarily giving up access to their money.

Loans are a type of investment. The lender can be thought to be investing the loan principle with the expectation of receiving a larger amount in return at some point in the future. Interest is also paid on other types of investments, such as bonds, which we will discuss in detail in later sections.

Accumulation and Amount Functions

In this class, we will use the concepts of accumulation and amount functions to track the growth rate and value of an investment that is earning interest. The definitions of these types of functions are provided in the box below.

Accumulation and Amount Functions

Assume that an amount of principal P is invested at time $t = 0$ in an account that earns interest.

- The **amount function** for the account is a function $A(t)$ that provides the value, or **balance**, of the account at time *t*. It follows that $A(0) = P$.
- The **accumulation function** for the account is a function $a(t)$ that provides the multiple by which the account has grown during the first *t* years.
- It follows from the definitions that $A(t)$ and $a(t)$ are related by $A(t) = P\,a(t)$.

There are two criteria that a function must satisfy to be a valid accumulation function. These are that $a(0) = 1$ and $a(t) > 0$ for all $t \geq 0$.

It is important to remember that an accumulation function $a(t)$ is required to satisfy the property $a(0) = 1$. We will see problems later in which we are required to find accumulation functions satisfying certain criteria. This property can help make this task easier, and also provides a way of double-checking our answers.

Example 1.1

An account earns interest according to the accumulation function $a(t) = 1 + 0.01\,t^2$.
Assume 100 is invested into the account at time 0.

a) Find the value of the account at the end of 2 years.
b) Find the value of the account at the end of 5 years

Example 1.2

A loan accumulates interest according to an accumulation function of the form $a(t) = e^{kt}$. The loan is to be repaid with a single payment of 2642.07 at the end of 10 years. An amount of 2078.33 was owed at the end of 6 years. Find the original loan amount.

We will see a wide variety of accumulation functions in this course, and most new concepts that we introduce early on will be explained in terms of general accumulation functions. That said, there are two specific forms of accumulation functions that will be of particular importance to us. Those are the accumulation functions associated with **simple interest** and **compound interest**. In simple interest, the account balance increases linearly over time, as opposed to compound interest where the account balance increases exponentially.

Simple Interest

Simple interest is a form of interest accumulation in which the value of the investment increases linearly over time. There are infinitely many simple interest accumulation functions, but they all take a similar form. Specific simple interest accumulation functions are defined by the choice for the value of a special constant *i*, called the simple interest rate.

Simple Interest

- Let *i* be a constant, which we will call the **annual simple interest rate**.
- The simple interest accumulation function defined by *i* is given by $a(t) = 1 + it$.

Since the accumulation function for simple interest is a linear function of *t*, the net growth in the account over any two time periods of equal length will be exactly the same.

Example 1.3

A loan of 200 is charged simple interest at an annual rate of 10%.

a) Find the amount owed at the end of 5 years.
b) Find the increase in the amount owed during year 5.
c) Find the increase in the amount owed during year 6.

Simple interest is generally applied when working with short-term loans or investments whose duration is less than one or two years. For longer-term investments, it is more standard to use compound interest, which we will discuss next.

Compound Interest (Annual Compounding)

Compound interest refers to any form of interest in which the accumulation is represented by an exponential function. Nearly all of the problems you will encounter in this course will use accumulation functions associated with compound interest, and you will be introduced to many alternate forms for such accumulation functions. We will begin by considering the most basic form of accumulation function associated with compound interest, interest with annual compounding.

Compound Interest (Annual Compounding)

- Let *i* be be a constant, which we will call the **annual effective interest rate**.
- The compound interest accumulation function defined by *i* is given by $a(t) = (1 + i)^t$.

Example 1.4

A loan of 200 is charged compound interest at an annual effective rate of 10%.

a) Find the amount owed at the end of 5 years.
b) Find the increase in the amount owed during year 5.
c) Find the increase in the amount owed during year 6.

As mentioned previously, most of the problems encountered in this course will deal with compound interest. The instruction to use compound interest is sometimes related by explicitly stating that an investment earns compound interest. In other cases, however, this information is conveyed in the way that the interest rate is described. Any time that you see an interest rate described using some variation of the words "annual" and "effective", you should assume that you to being asked to use compound interest.

We will now give several examples of sentences that can be used to indicate the use of compound interest accumulation functions. The following statements all mean exactly the same thing.

- A loan is charged compound interest at an annual effective rate of 6%.
- A loan is charged interest at an annual effective rate of 6%.
- A loan is charged interest at a rate of 6% annual effective.
- A loan is charged interest at an effective rate of 6% per annum.
- A loan is charged interest at a rate of 6% compounded annually.

Example 1.5

Erlich simultaneously makes investments into two different funds: Fund A, and Fund B.

- Fund A earns interest at a rate of 4% annual effective.
- Fund B earns interest at a 6% annual effective rate.
- The amount invested in Fund A was twice the amount invested in Fund B.
- After 6 years, the two accounts are collectively worth $20,000.

Determine the combined value of the funds 10 years after the initial investments.

Example 1.6

Lucy and Patty each invest 200. Lucy's investment earns compound interest and Patty's investment earns simple interest. At the end of 2 years, the two investments are of equal value. At the end of 4 years, the value of Lucy's investment is 1.05 times that of Patty's investment. Find the value of Patty's investment after 5 years.

The motivation behind the formula for annual compounding is the assumption that interest is applied to the account at the end of each year. The interest accumulated each year is added to the current balance, and will itself earn interest during the subsequent years. As a result, the value of the account will increase by a factor of $1+i$ each year. After t years, the value will have increased by a factor of $a(t) = (1+i)^t$.

This interpretation implies that the balance is constant throughout the year, only increasing at the end of the year. This would technically result in a piece-wise defined accumulation function with jump discontinuities at the end of each year when the compounding occurs. To avoid complications stemming from working with discontinuous functions, we will assume that the balance is continuously increasing and that the the accumulation function $a(t) = (1+i)^t$ is valid for all times t.

Example 1.7

Beth deposits 200 into an account that earns a 4% annual effective rate of interest. Jerry deposits 300 into an account that earns interest at an effective rate of 8% per annum. How many years pass until the two accounts have the same balance? Round your answer to 2 decimal places.

Compound Interest (Continuous Compounding)

As mentioned earlier, there are other types of accumulation functions that can be used to represent compound interest. The choice of which form to use depends upon how the rate you are provided has been described. We already know that when we are told that the rate is an "annual effective rate", then you would use the accumulation function $a(t) = (1+i)^t$, where i denotes the rate. If, however, you are told that the rate is **compounded continuously**, then you would use an accumulation function of the form described below.

Compound Interest (Continuous Compounding)

- Let δ be be a constant, which we will call either the **continuously compounded rate of interest** or the **force of interest**.
- The compound interest accumulation function defined by δ is given by $a(t) = e^{\delta t}$.

Although we will tend to use δ to denote the continuously compounded rate of interest, it is also quite common to see r used to denote this rate.

Example 1.8 A fund earns interest at a continuously compounded rate of 12%. How long does it take for the value of the fund to double?

We have introduced two quite different-looking functions for compound interest, $a(t) = (1+i)^t$ and $a(t) = e^{\delta t}$. The forms for these accumulation functions are not as different as they might seem at first glance. They are, in fact, simply different expressions of the same idea. Any function of the form $a(t) = (1+i)^t$ can also be written in the form $a(t) = e^{\delta t}$, although the values of i and δ will differ slightly.

It might seem redundant to introduce two different ways of expressing the same concept, but both of these forms are useful. Each form has situations for which it is more convenient to use than the alternate form.

Example 1.9 Assume that $\delta = 8\%$ and $(1+i)^t = e^{\delta t}$. Find i.

The rates δ and i are said to be equivalent rates since the accumulation functions associated with them are the same, and thus they describe the same rates of growth (although in slightly different ways). We will discuss equivalent rates in more detail in Section 1.5.

Compound Interest: Varying Annual Rates

It is not uncommon for an account earning compound interest to earn different effective rates during different years. The process for dealing with such scenarios is explained below.

- Assume an account earns an effective annual rate of i_1 during the first year, i_2 during the second year, and so on, eventually earning a rate of i_t during year *t*.
- The accumulation function for the account is given by $a(t) = (1+i_1)\cdot(1+i_2)\cdot\ldots\cdot(1+i_t)$.

Example 1.10 Joe deposits 100 at time 0. His account earns 5% in year 1, 7% in year 2, and 4% in year 3. Find the value of the account at the end of the third year.

Amount of Interest Earned

Assume that a certain amout of principal is invested into an account at time 0, and no other payments are made into the account. The amount of interest earned by the account during a specific time period is equal to the increase in the value of the account during that time period. More precisely, the amount of interest earned during the period between times t_1 and t_2 is equal to $A(t_2) - A(t_1)$.

Example 1.11 A loan of 750 is charged interest according to the accumulation function $a(t) = 1 + 0.08t^2$. Determine the amount of interest accumulated between the end of the 16th and 24th months.

We are often asked to determine the amount of interest earned on an investment during a specific one year period. Let I_t denote the interest accumulated during year *t*, where *t* is a whole number. Since year *t* starts at time $t-1$ and ends at time t, we get that $I_t = A(t) - A(t-1)$.

We can apply the formula $I_t = A(t) - A(t-1)$ when working with any sort of amount function. But since we will predominately work with simple and compound interest, it is useful to have special formulas for I_t to apply when we are working specifically with these types of interest. Such formulas are provided below. There derivations are not show, but they following directly from the expression $I_t = A(t) - A(t-1)$.

Amount of Interest Earned During Year *t*

Assume a deposit of P is made into an account earning interest according to some accumulation function. Let I_t refer to the amount of interest earned during year *t*.

- In general, we have that $I_t = A(t) - A(t-1)$.
- If the account earns simple interest at an annual rate of *i*, then $I_t = i \cdot P$.
- If the account earns compound interest at an annual effective rate of *i*, then $I_t = i \cdot A(t-1)$.

Example 1.12 Steven deposits 500 into Fund X and 500 into Fund Y at time 0. Fund X earns simple interest at an annual rate of 6%. Fund Y earns compound interest at an annual effective rate of 6%. Calculate the total amount of interest that Steven earns during the fourth year.

1.2 EFFECTIVE RATES OF INTEREST

In the previous section, we defined the concept of an "annual effective rate of interest". Such an interest rate was defined specifically for use when working with compound interest. It turns out, however, that we can generalize the term "effective rate of interest" in ways that allow it to be applied to situations involving any accumulation functions, not just those associated with compound interest. We will also discuss effective rates of interest over non-annual periods.

Annual Effective Rate For a Given Year

Assume that an investment grows according to some accumulation function $a(t)$. The **annual effective rate of interest during a given year** measures the percentage growth in an account during that specific year. We will denote the effective annual rate during year t by i_t. It is important to note that year t runs from time $t-1$ to time t. Formulas for i_t are provided below.

Annual Effective Rate During Year *t*

- Assume that an investment grows according to an accumulation function $a(t)$.
- The annual effective rate of interest during year t is given by: $i_t = \dfrac{A(t) - A(t-1)}{A(t-1)}$.
- It can be shown that i_t can also be calculated using the formula: $i_t = \dfrac{a(t) - a(t-1)}{a(t-1)}$.

Example 1.13 An account earns interest according to the accumulation function $a(t) = e^{0.01t^2}$.

a) Find i_6, the annual effective rate of interest during year 6.
b) Find i_7, the annual effective rate of interest during year 7.
c) Find i_8, the annual effective rate of interest during year 8.

Assume that an account earns compound interest at an annual effective rate of i. It can be shown that i_t is constant for such an account, and in fact $i_t = i$ for any t. This illustrates that the phrase "annual effective rate" is consistent between the definition presented here and the one provided for compound interest. In fact, i_t can be thought of as the rate of growth in an account over a one year period *assuming* that growth was the result of accumulation by compound interest.

Although i_t is constant in the case of compound interest, it is a decreasing function of t when working with simple interest. This fact is demonstrated in the following example.

Example 1.14 An account earns simple interest at an annual rate of 10%.

a) Show that $i_t = \dfrac{0.1}{0.1t + 0.9}$.
b) Find i_t during each of the first five years.
c) During what year is i_t first less than 5%?

Calculator Tip: Parts (b) and (c) in Example 1.14 can be easily solved using the "table" function of the TI-30XS calculator. I recommend reading about this function in the calculator manual. It can be very useful.

Annual Effective Rate Over a Period

Assume that an investment grows according to an accumulation function $a(t)$. The **annual effective rate of interest during a *t* year period** is defined to be the equal to the annual effective rate of compound interest that would have produced the same amount of growth during that same period of time.

Annual Effective Rate During a *t* Year Period

- Assume that an investment grows according to an accumulation function $a(t)$.
- The annual effective rate of interest during the first t years can be found by solving for i in the following equation: $(1+i)^t = a(t)$.

The annual effective rate of interest during a period is found by assuming that total amount of growth seen during that period was the result of compound interest.

Example 1.15 An account earns interest according to the accumulation function $a(t) = e^{0.01t^2}$.

a) Find the annual effective rate of interest over the course of the first 4 years.
b) Find the annual effective rate of interest over the course of the first 8 years.

Example 1.16 An account earns simple interest at an annual rate of 10%.

a) Find the annual effective rate of interest during the first 5 years.
b) Find the annual effective rate of interest during the first 10 years.

Compare the results of the next example to the answer in Example 1.9.

Example 1.17 An account earns compound interest continuously at a rate of 8%.

c) Find the annual effective rate of interest during the first 5 years.
d) Find the annual effective rate of interest during the first 10 years.

The annual effective rate over a period of several years can be thought of as a sort of average of the annual effective rates for each of the years during that period. The average is not, however, a standard arithmetic average, as illustrated in the next example.

Example 1.18 An account earns 5% in year 1, 3% in year 2, and 7% in year 3. Find the annual effective rate of interest earned over the course of the three years.

1.3 PRESENT VALUE AND CURRENT VALUE

Present Value

As mentioned in Section 1.1, the inherent value of a payment depends in part on when the payment is received. If we would like to compare the value of two payments occurring at different times, then we need to select a specific time and use an accumulation function to determine the value of each of the two payments at this particular time. As we will see later, the particular time t at which we value the payments is somewhat arbitrary. However, it is often convenient to determine the time value of the two payments at time $t=0$. The time value of a payment at time $t=0$ is called the **present value (PV)** of the payment.

Present Value (PV) for General Accumulation Functions

- Assume you are provided an accumulation function $a(t)$.
- The present value of a payment of 1 occurring at time t is given by $PV = \dfrac{1}{a(t)}$.
- The present value of a payment of K occurring at time t is given by $PV = \dfrac{K}{a(t)}$.

Let PV be the the present value of a payment K due at time t. The value PV can be interpreted in several ways:

- An investment of PV at time 0 would grow to an amount of K at time t under the effects of $a(t)$.
- If a loan of PV is made at time 0 and accumulates interest according to the accumulation function $a(t)$, then K is the amount that will be owed to the lender at time t.
- According to the accumulation function $a(t)$, a payment of PV at time 0 has the same time value as a payment of K at time t.
- If an individual can readily borrow and lend money that will accumulate interest according to the accumulation function $a(t)$, then they will (in theory) place the same value on a payment of PV at time 0 as they would a payment of K at time t.

Example 1.19 Given the accumulation function $a(t) = (1 + 0.1t)^2$, find the present value of a payment of 1000 at the end of four years. In other words, determine the amount that would need to be invested at time 0 in order for the investment to grow to 1000 at the end of four years.

The process of "moving a payment back in time" in order to find its present value is called **discounting**. Recall that the process of "moving a payment forward in time" to find its future value is called **accumulating**.

Present Value for Compound Interest

Note that in order to accumulate forward t years under compound interest, then we simply multiply the original principal by a certain accumulation factor t times. The accumulation factor is equal to $(i+i)$ in the case of annual compounding and $e^{\delta t}$ for continuous compounding. It follows that if we want to discount a payment at time t back to time 0, then we simply need to divide by this factor t times. This is equivalent to multiplying by the reciprocal of the accumulation factor t times. When working with compound interest, we will refer to the reciprocal of an accumulation factor as a **present value factor**.

Present Value (PV) for Compound Interest

- Assume an account earns compound interest (either annually, or continuously). We will define the **present value factor**, denoted by v, as follows:
  - **Annual Compounding:** Let $v = \dfrac{1}{1+i}$.
  - **Continuous Compounding:** Let $v = e^{-\delta}$.
- In either case, the PV of a payment of K at time t is given by $PV = K v^t$.

When working with compound interest, it is important to remember the following facts:

- Multiplying the value of a payment by an accumulation factor $(1+i)$ or e^{δ} accumulates by one year, or in other words, determines the time value of the payment one year in the future. Multiplying by the accumulation factor n times carries the value of the payment forward n years.
- Multiplying the value of a payment by the present value factor v discounts the payment by one year. That is, it determines the time value of the payment at a time one year earlier. Multiplying by the present value factor n times discounts the payment by n years.

Example 1.20 Assuming an annual effective interest rate of 8%, which of the following payments has a larger present value: A payment of 120 at time 4, or a payment of 150 at time 7?

Calculator Tip: It is often the case that you will need to use the same present value factor multiple times in the same problem. For such problems, it can be convenient first calculate v, and then store it in one of the calculator registers, such as x. After doing so, if you wanted to calculate the PV of, for example, a payment of 70 at time 8, you would enter: $70\,x^8$.

Accumulating Over General Periods (The Delayed Deposit Trap)

Recall that $a(t)$ was said to provide the factor by which an account grows in the t years immediately after depositing the initial deposit or loan. This statement assumes that the initial transaction took place at time 0. In other words, $a(t)$ does not provide the accumulation factor for an arbitrary t year period, but instead one that begins at time 0.

Assume $t_1 < t_2$. Given an accumulation function $a(t)$, if we wish to accumulate a payment of P occurring at time t_1 forward to time t_2, we first discount the payment to time 0 to obtain a present value of $P / a(t_1)$. We then accumulate this PV forward to time t_2 to obtain an accumulated value of $P \cdot a(t_2) / a(t_1)$. The factor $a(t_2) / a(t_1)$ describes the growth during the period $t \in [t_1, t_2]$. We will denote this quantity by $a(t_1, t_2)$.

Accumulating Over General Periods

- Let $a(t)$ be an accumulate function and let $t_1 < t_2$.
- Define $a(t_1, t_2)$ by $a(t_1, t_2) = \dfrac{a(t_2)}{a(t_1)}$.
- Assume an account has a value of P at time t_1. The accumulated value of this account at time t_2 is given by $P \cdot a(t_1, t_2)$.

Example 1.21 The value of an account grows according to the accumulation function $a(t) = (1 + 0.1t)^2$. The account is worth 2000 at the end of the year 3. Find the value of the account at the end of year 7.

Example 1.22 A fund grows according to the accumulation function $a(t) = 1 + 0.05t^2$. A deposit of X is made into the account at time 6. Determine the number of years required for the value of the deposit to double.

Note that the situation is much simpler when working with compound interest. It can be shown that when working with compound interest, the accumulation factor during ANY t year period will be equal to $(1 + i)^t$. For all other accumulation functions, one must apply the accumulation factor $a(t_1, t_2) = a(t_2) / a(t_1)$ when accumulating over a period that does not begin at time 0.

Example 1.23 The value of an account accumulates interest an an annual effective rate of 7%. The account is worth 2000 at the end of the year 3. Find the value of the account at the end of year 7.

Present Value at a Specified Time (Current Value)

It is often necessary to discount a payment to a time other than 0. Assume that $t_1 < t_2$. To discount a payment of *K* occurring at time t_2 back to time t_1, we divide *K* by the accumulation factor $a(t_1, t_2)$ to obtain a discounted value of $K \cdot a(t_1) / a(t_2)$. This discounted value is referred to as the **present (or current) value at time t_1** of the payment, and can be interpreted as the amount that must be invested at time t_1 for the account to be worth *K* at time t_2.

Present Value at a Specific Time

- Let $a(t)$ be an accumulate function and let $t_1 < t_2$.
- Define $a(t_1, t_2)$ by $a(t_1, t_2) = \dfrac{a(t_2)}{a(t_1)}$.
- The present (or current) value at time t_1 of a payment of *K* occurring at time t_2 is given by $PV = \dfrac{K}{a(t_1, t_2)} = K \cdot \dfrac{a(t_1)}{a(t_2)}$.

Any mention of present value that does not specify a particular time will refer to the present value at time 0.

Example 1.24 Given the accumulation function $a(t) = 1 + 0.05t^2$, find the present value at time 4 of a payment of 3250 occurring at time 10.

Flexibility of $a(t_1, t_2)$

Up to this point, we have only use the function $a(t_1, t_2)$ in situations where $t_1 < t_2$.

Assume that $t_1 < t_2$. We have established the following two facts in this section:

- To accumulate a payment from time t_1 to time t_2, we multiply the payment by $a(t_2) / a(t_1)$.
- To discount a payment from time t_2 to time t_1, we multiply the payment by $a(t_1) / a(t_2)$.

In either case, we are multiplying by a factor of the form $a(t) / a(s)$, where *s* is the time of payment, and *t* is the time that we are moving the payment to. We can unify these two rules into the following rule:

- Assume a payment of *K* occurs at time *s*. The value of the payment at time *t* equal $K \cdot a(s, t) = K \cdot \dfrac{a(t)}{a(s)}$.

This new rules works whether we are accumulating or discounting.

Equivalent payments

It can be shown that if two payments have the same present value at a certain time t_1, then they will have the same present value at ALL times *t*. As a result, we consider the two payments to be **equivalent** with respect to the accumulation function being used.

1.4 PRESENT VALUE OF A SEQUENCE AND METHOD OF EQUATED TIME

Present Value of a Sequence of Payments

We will define the present value of a sequence of payments to be equal to the sum of the present values of each of the individual payments. A single payment of this amount at time 0 is considered to have the same time value as the entire sequence of payments.

Present Value of a Sequence of Payments

- Let K_1, K_2, …, K_n be the value of payments made at times t_1, t_2, …, t_n, respectively.
- The present value of the payments is given by $PV = \frac{K_1}{a(t_1)} + \frac{K_2}{a(t_2)} + ... + \frac{K_n}{a(t_n)}$.
- Assuming compound interest, then we have that $PV = K_1 v^{t_1} + K_2 v^{t_2} + ... + K_n v^{t_n}$.

Example 1.25 Assume an annual effective interest rate of 5%. Find the present value of the following sequence of payments: 100 at time 2, 300 at time 4, and 250 at time 5.

The answer in Example 1.25 is 533.40. There are several useful ways in which one can interpret this value:

- Assume that a loan of 533.40 is made at time 0 at an annual effective rate of 5%. If the borrower makes payments of 100 at time 2, 300 at time 4, and 250 at time 5, then the loan will be completely paid off after the last payment. It should be noted that this is but one of many possible ways to repay such a loan. Any sequence of payments with the same present value would represent a valid repayment plan for this loan.
- Assume that 533.40 is invested at time 0 at an annual effective rate of 5%. This deposit could fund withdrawals of 100 at time 2, 300 at time 4, and 250 at time 5. After the withdrawal at time 5, the account would be empty.
- Assuming an annual effective interest rate of 5%, a payment of 533.40 has exactly the same time value as the entire sequence of payments described in the problem.

Example 1.26 At an annual effective interest rate of *i*, the following two sets of payments described below have the same present value, *P*. Find *P*.

i) A payment of 140 at the end of year one, and a payment of 140 at the end of year four.
ii) A payment of 200 at the end of year three, and a payment of 200 at the end of year six.

We will occasionally encounter problems in which we are asked to consider two or more possible payment plans for a loan. Such a situation could occur if the borrower misses payments and thus requires a new repayment plan to be developed, or if the borrower decides to repay the loan early. The key to solving such problems is to remember that any sequence of payments whose present value is equal to the loan amount will represent a valid repayment plan.

Example 1.27 Elbert borrows 2400 at an annual effective interest rate of i. He intends to repay the loan by making a payment of 1800 after three years, and another payment of 1327.63 after six years.

Elbert pays the first payment as normal. At the end of the fourth year, Elbert decides to repay the loan by paying off the remaining balance of the loan, which is equal to K. Find K.

It is possible that when renegotiating a loan, there will be additional penalties incurred by the party wishing to renegotiate. Such penalties can have an effect on the annual effective rate ultimately realized by the loan.

Example 1.28 Assume that in the previous example, Elbert is required to pay an early payment penalty of 50 along with his second payment of K. Use the **table** function of the TI-30XS to estimate the annual effective interest rate actual paid by Elbert when the penalty is taken into account. Round your answer to three decimal places.

Example 1.29 An account earns interest at an annual effective interest rate of i. Deposits are made into the account as follows: 200 is deposited at time 0, 300 is deposited at time n, and 500 is deposited at time $2n$. The accumulated value of the fund at the end of the tenth year is 1512.12.

Given that $v^n = 0.8209$, find i.

Present Value of a Sequence of Payments at Time t_1

As with individual payments, it is possible to value a sequence of payments at any given time t_1. Doing so would produce the value of a payment that, if paid at time t_1, would have the same time value as the given sequence. There are two equivalent methods we can use to find the present (or current) value of a sequence at time at time t_1.

Present Value of a Sequence of Payments at Time t_1

The present (or current) value at time t_1 of a sequence of payments can be found by using either of the following (equivalent) methods:

1. Calculate the present value of the sequence at time 0. Accumulate the resulting present value forward to time t_1.
2. Calculate the current value at t_1 of each of the payments separately, and then sum the results. Payments occurring prior to time t_1 will need to be accumulated forward, whereas payments occurring after time t_1 will need to be discounted backward in time.

It is important to know that the two methods above produce equivalent results. However, I would generally recommend using the second method to calculate current value of a sequence.

If two sequences of payments have the same present value at time 0 (and thus at all times t) then we will consider them to be equivalent. Equivalent sequences will have the same time value. Two equivalent sequences can be viewed as both being valid repayment plans for the same loan.

Example 1.30 Dana and Ed each take out loans of the same size. The loans collect interest at an effective annual rate of 4.5%. Dana plans to pay off her loan by making a payments of 100 at the end of years n, $n+1$, $n+2$, and $n+3$. Ed agrees to repay his loan with a single payment of P at the end of year $n+1$. Find P.

Example 1.31 Morty borrows an amount of L at time 0. The loan is charged interest at an annual effective rate of 5%. According to the original terms of the loan, Morty is required to make payments of 100 at the end of each of the first four years, at which point the loan will be repaid.

Morty instead pays the loan off early by making a payment of 100 at the end of year 1, 125 at the end of year 2, and a payment of X at the end of year 3. Find X.

Method of Equated Time

Assume that n payments of $P_1, P_2, \ldots, P_n$ are made at times $t_1, t_2, \ldots, t_n$, respectively. The **method of equated time** provides an estimate $\bar{t}$ of the time at which a single payment of $P_1 + P_2 + \ldots + P_n$ has same present value as the original series of payments. The estimate $\bar{t}$ is calculated as a weighted average of the times at which the payments are made, with the weights provided by the size of the payments.

Method of Equated Time

- Assume that n payments of $P_1, P_2, \ldots, P_n$ are made at times $t_1, t_2, \ldots, t_n$, respectively.
- Let $\bar{t} = \dfrac{P_1 t_1 + P_2 t_2 + \ldots + P_n t_n}{P_1 + P_2 + \ldots + P_n}$.
- The time $\bar{t}$ provides an approximation for the time at which a payment of $P_1+P_2+\ldots+P_n$ would be equivalent to the given sequence.

Note that $\bar{t}$ is just an estimate. If we want to know the exact time t at which as single payment in an amount of $P_1 + P_2 + \ldots + P_n$ has the same present as the original series of payments, then we would need to solve the equation $(P_1 + P_2 + \ldots + P_n)v^t = P_1 v^{t_1} + P_2 v^{t_2} + \ldots + P_n v^{t_n}$.

Example 1.32 Payments of 300, 500, and 200 are made at the end of years two, four, and seven, respectively.

a) Use the method of equated time to estimate the time at which a single payment of 1000 would be equivalent to the described sequence of payments.

b) Assuming an annual effective interest rate of 6%, determine the actual time when a single payment of 1000 would be equivalent to the given sequence.

Example 1.33 John borrows P at an annual effective rate of 5%. He agrees to repay the loan by making payments of 200 at the end of year 1, 250 at the end of year 2, and X at the end of year 4.

Using the method of equated time, we determine that John could have also repaid the loan by making a payment of $450 + X$ at approximately time $t = 2.625$.

Find X.

1.5 NOMINAL RATES OF INTEREST

Effective Periodic Rates

Up to this point, we have placed an emphasis on using years as a unit of time. Annual effective rates measure the growth over the course of a year, and our time variable t is always measured in years. Although years are generally convenient to work with, there is no reason why a year must be the default unit of time. There are many types of investments that will accumulate interest every six months, as well as those that accumulate interest every month. When working with such types of investments, it may make sense to use six month or one month periods as our default unit of time. If we are using something other than years as our default unit of time, then it is usually convenient to work with non-annual effective rates that measure the growth over such a period of time.

Effective Periodic Rates for Compound Interest

- Define an ***m*-thly period** to be a length of time equal to $1/m$ years.
- There are m such periods during each year.
- Assume an account earns compound interest.
- We will define the **effective *m*-thly rate** to be the rate of growth for the account during each m-thly period. When the length of the period is clear, then we sometimes refer to this rate as the **effective periodic rate**.
- Assume that P is invested into an account earning compound interest at an m-thly effective rate of j. The value of the account after N periods is $P(1+j)^N$

As hinted at in the box above, we will generally use j (or sometimes k) to refer to an m-thly effective rate, reserving i specifically for annual effective rates.

We assign special names to m-thly rates for certain values of m:
- If $m = 2$, we will refer to an effective m-thly rate as an **effective semi-annual rate**.
- If $m = 4$, we will refer to an effective m-thly effective rate as an **effective quarterly rate**.
- If $m = 12$, we will refer to an effective m-thly effective rate as an **effective monthly rate**.

Example 1.34 Assume an account earns compound interest at an effective monthly rate of 1%. An amount of 250 is deposited into an account. Find the value of the account after 1.5 years.

Nominal Rates of Interest

It is standard practice when stating an m-thly rates to scale the rate to a full year by multiplying it by m. Such a rate is called a **nominal annual rate compounded (or convertible) *m*-thly**, and is denoted by $i^{(m)}$. As we will soon see, $i^{(m)}$ is something different from an annual effective rate. We can calculate the annual effective rate for a problem involving nominal rates of interest, but the rate i will differ slightly from the value of $i^{(m)}$.

When encountering a problem that uses a nominal rate $i^{(m)}$, first step is almost always to divide the nominal rate by m to obtain an effective periodic rate, and then move forward working on an m-thly basis.

Nominal Annual Rates of Interest

- The **nominal annual interest rate compounded (or convertible) *m*-thly** is denoted by $i^{(m)}$.
- Given $i^{(m)}$, the effective periodic rate is given by $j = \dfrac{i^{(m)}}{m}$.
- The effective annual rate i can be found by solving the equation $(1+i) = (1+j)^m$.

It is important to remember that nominal rates are simply a means of indirectly reporting the effective periodic rates. Nominal rates should almost never be used directly to calculate anything other than j.

Pay very careful attention to whether annual rates provided to you in a problem are nominal or effective. A very common mistake is to treat a nominal rate as an effective annual rate, or vice versa.

Example 1.35 Assume an account earns compound interest at an effective monthly rate of 1%.
a) Find the nominal annual rate, $i^{(12)}$.
b) Find the annual effective rate, i.

Example 1.36 A loan of 3500 collects interest at a nominal rate of $i^{(4)} = 5\%$. The loan is repaid with a single payment after 28 months. Find the size of the payment.

When asked to determine a nominal rate, you should start by finding the effective periodic rate and then scaling to a nominal annual rate by multiplying by *m*.

Example 1.37 A loan of 2400 collects interest at a nominal rate of $i^{(12)} = r\%$. The loan is repaid with a single payment of 3085.12 after 3 years. Find *r*.

Example 1.38 Joe and Mick simultaneously make investments into different funds. You are given the following information about the investments:

- Joe's fund earns interest at a nominal rate of 8% convertible quarterly.
- Mick's fund earns interest at a nominal rate of 6% compounded every six months.
- Joe invests 50 more than Mick.
- At the end of 5 years, the value of Joe's fund is twice the value of Mick's fund.

Determine the size of Mick's initial investment.

Although we will generally reserve the symbol *i* to refer to an annual effective rate, it does occasionally show up in problems in reference to nominal rates. It is important to read the problems carefully to make sure you understand what type of rates are being used.

Example 1.39 Steve and Tony each make investments at the same time. Steve invests *K* into a fund that accumulates interest at a nominal annual rate of *i*, convertible quarterly. Tony invests 2*K* into a fund that earns simple interest at an annual rate of *i*. Steve and Tony earn the same amount of interest during the last quarter of the sixth year. Find *i*.

Equivalent Rates

We have seen several ways of expressing interest rates for compound interest problems. We have worked with annual effective rates, continuously compounded rates, and nominal annual rates. These different types of rates are simply different ways of expressing the same type of growth. Every nominal rate has a unique annual effective rate that will produce the same growth. There will also be a unique continuously compounded rate that produces the same growth.

We say that two rates are **equivalent** if they have the same annual effective rate. Equivalent rates will produce the same accumulation factor over periods of equal length. This observation is useful for finding equivalent rates. For instance, a nominal rate $i^{(2)}$ and a continuously compounded rate δ are equivalent if they produce the same accumulation over the course of one year, and thus satisfy the equation: $\left(1 + i^{(2)} / 2\right)^2 = e^{\delta}$

As an example, consider the nominal annual rate $i^{(2)} = 6\%$. This nominal rate corresponds to an effective semi-annual rate of $j = 3\%$. The equivalent annual effective rate can be found by solving $(1.03)^2 = 1+i$, which yields $i = 6.09\%$. We can calculate the force of interest to be $\delta = 5.9118\%$ by solving the equation $(1.03)^2 = e^{\delta t}$. The rates $i^{(2)} = 6\%$, $j = 3\%$, $i = 6.09\%$, and $\delta = 5.9118\%$ all describe the same rate of growth, and are thus considered to be equivalent.

Example 1.40 Fund A collects interest at a nominal annual rate of 10% convertible quarterly. Fund B collects interest at a nominal annual rate of *r*% convertible semiannually. The rates for the two funds are equivalent. Find *r*.

It can seem overwhelming to work with so many different types of rates when they are just different ways of expressing the same thing. However, each type of rate we have considered has practical uses.

Compounding Periods Less Frequent than One Year

It is possible (though rare) to encounter compound interest problems in which interest is compounded less frequently than once a year. For example, a problem might state that interest is compounded once every two years, every three years, etc. In such problems, the rate will generally be expressed as an nominal annual rate in which the rate has been scaled down to a year. Although this notation is nonstandard, it would be consistent to denote a nominal rate for an account that compounds interest every M years as $i^{(1/M)}$.

For instance, assume that an account accumulates interest at a nominal annual rate of 5%, compounded every three years. Then $i^{(1/3)} = 5\%$. The effective three-year rate would then be $j = 15\%$. The annual effective rate could be found by solving the equation $(1 + i)^3 = (1.15)$. Doing so yields $i = 4.769\%$.

Example 1.41 An account earns interest at a nominal annual rate of $i^{(1/2)} = 10\%$ compounded every 2 years.

a) Find the effective 2 year rate.
b) Find the value of an investment of 100 after six years.
c) Find the value of an investment of 100 after three years.
d) Find the effective annual rate.
e) Find an equivalent nominal semiannual rate.

Example 1.42

The interest rate on a four year investment varies from year to year as follows:

i) During year 1, the fund earns an effective annual rate of 7%.
ii) During year 2, the fund earns a continuously compounded rate of 8%.
iii) During year 3, the fund earns a nominal annual rate of 9% compounded semiannually.
iv) During year 4, the fund earns a nominal annual rate of 5% convertible every two years.

Determine the effective annual rate of interest over the four year period.

1.6 RATES OF DISCOUNT

Rates of Discount

We have already seen several different methods of representing a rate of compound interest. In this section, we will learn about one more type of rate, called a **discount rate**.

When calculating the percentage difference between two values, we will obtain different results depending on which of the two values we use as the "base" value. For example, we can say that 150 is 50% more than 100, but it would also be correct to say that 100 is (roughly) 33% less than 150. Put another way:

- Changing from 100 to 150 represents a 50% increase.
- Changing from 150 to 100 represents a 33% decrease.

Whereas interest rates are rates of increase, discount rates represent rates of decrease.

Rates of Discount

- Assume an account grows according the the accumulation function $a(t)$.
- Recall that the annual effective rate of interest during year *t* is given by $i_t = \dfrac{a(t) - a(t-1)}{a(t-1)}$.
- The annual effective rate of discount during year *t* is defined as $d_t = \dfrac{a(t) - a(t-1)}{a(t)}$.

Example 1.43 Assume a fund is worth 80 at the beginning of year three and is worth 100 at the end of year three. Find the effective annual rates of interest and discount during year three.

Discount Rates For Compound Interest

We have seen in the past that the annual effective rate of interest i_t can vary over time when working with a general accumulation function $a(t)$. The same is true for discount rates d_t. However, when working with compound interest, both of these rates are constant. As usual, let *i* represent this constant annual effective rate of interest. We will now use *d* to denote the constant **annual effective rate of discount**. There are many useful relationships between *i*, *d*, and *v*.

Using the formulas above, we can derive a relationship for between the rates *i* and *d*. The derivation is as follows:

$$d = \frac{(1+i)^t - (1+i)^{t-1}}{(1+i)^t} = 1 - \frac{1}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} = \frac{1+i-1}{1+i} = \frac{i}{1+i}$$

One of the most important aspects of the discount rate is its relationship with the present value factor, *v*. A present value factor of $v = 0.97$ corresponds to an interest rate of $i = 0.0309278$ (verify this on your own). We can use the formula above to determine that the associated discount rate is $d = 0.03$. This hits that *d* and *v* are related by the expression $v = 1 - d$. We can verify this fact as follows:

$$1 - d = 1 - \frac{i}{1+i} = \frac{1+i}{1+i} - \frac{i}{1+i} = \frac{1+i-i}{1+i} = \frac{1}{1+i} = v$$

We will now summarize what we know about discount rates for compound interest. We will also state a few identities that we have not yet discussed, but can be useful at times.

Identities Involving Discount Rates

The following identities hold true for *i*, *d*, and *v*.

- $i = \dfrac{d}{1-d}$
- $d = \dfrac{i}{1+i}$
- $v = 1 - d$
- $d = i\,v$
- $i - d = i\,d$
- $\dfrac{1}{d} - \dfrac{1}{i} = 1$

Example 1.44 Find the annual effective interest rate *i* that is equivalent to an annual effective discount rate of $d = 8\%$. Also calculate the associated present value factor, *v*.

Accumulating With Discount Rates

The expression $v = 1 - d$ tells us how to convert easily between discount rates and present value factors, but it also gives us a means of accumulating using discount rates. If $v = 1 - d$, then $v^t = (1-d)^t$. It then follows that $a(t) = (1+i)^t = 1 / v^t = v^{-t} = (1-d)^{-t}$. We state this result along with some related results below.

Accumulating and Discounting

Assume that an account earns compound interest at an annual effective rate of *i*. Let *d* be the equivalent rate of discount.

- The accumulation function for the account can be written in either of the following forms
  $a(t) = (1+i)^t$ or $a(t) = (1-d)^{-t}$
- To accumulate a payment forward one year, we can **multiply** it by $1+i$ or **divide** it by $1-d$.
- To discount a payment back one year, we can **divide** it by $1+i$ or **multiply** by $1-d$.

Example 1.45 Find the present value of the following sequence of payments: A payment of 300 at the end of year 1, a payment of 500 at the end of year 3, and a payment of 200 at the end of year 5. Assume an annual effective discount rate of $d = 6\%$.

Example 1.46 An account earns a 4% annual effective rate of interest during year 1 and a 7% annual effective rate of discount during year 2. Find the effective annual rate of interest during the 2 year period.

Nominal Discount Rates

As with interest rates, discount rates can be expressed in terms of a non-annual period. Such non-annual periodic discount rates are stated in the form of scaled **nominal annual discount rates.**

Nominal Annual Rates of Discount

- The **nominal annual discount rate compounded (or convertible) *m*-thly** is denoted by $d^{(m)}$.
- Given $d^{(m)}$, the effective periodic rate is given by $k = \dfrac{d^{(m)}}{m}$.
- The effective annual rate i can be found by solving the equation $(1+i) = (1-k)^{-m}$.

Example 1.47 Helga invests 5000 into an account. The account earns a nominal annual discount rate of 8% convertible quarterly during the first three years, and a nominal annual discount rate of 6% convertible quarterly during all later years.

a) Determine the value of the account at the end of 8 years.
b) Determine the annual effective rate of interest earned by the account during the first 8 years.

Example 1.48 An investment of 1000 is made to a fund. The fund earns a nominal annual discount rate of r convertible quarterly during the first year, and a nominal annual interest rate of r during the second year. The value of the account at the end of two years is 1185.34. Find r.

Simple Discount Rates

It is also possible to define the concept of simple discount rates. With simple interest, we assume that the accumulation factor scales linearly with time. With simple discount, we assume that it is, instead, the discount factor that scales linearly with time. For example, a simple discount rate of 3% would result in a one year discount factor of 0.97, a two year discount factor of 0.94, a three year discount factor of 0.91, and so on. The discount factor will decrease by 0.03 for each additional year that you wish to discount.

Rates of Simple Discount

- Let d be an annual rate of simple discount.
- The simple discount accumulation function defined by d is given by $a(t) = (1 - dt)^{-1}$.

Example 1.49 An amount of P is loaned at a simple discount rate of 7%. The loan is repaid after 3 years with a single payment of 1000. Find P.

It should be noted that while annual effective interest rates and annual effective discount rates for compound interest are two ways of expressing the exact same idea, the concepts of simple interest and simple discount are quite different from one another.

1.7 ZERO-COUPON BONDS AND T-BILLS

Bonds

It is often necessary for corporations and governments (both local and federal) to borrow money in order to fund projects. A corporation might need to borrow money in order to develop a new product, to build a new office building, to acquire another company, or any number of other reasons. Governments often need to borrow money to pay for various social programs, or to fund new infrastructure projects, such as building roads, bridges, airports, and stadiums. When a corporation or government needs to borrow large amounts of money, they generally borrow that money from the public in the form of **bonds**. The entity will split the amount they wish to borrow into many individual bonds, which investors can purchase, thereby making a small loan to the entity. When the bonds are issued, they include terms that describe how the debt will be repaid to the investor.

Assume that a corporation needs to raise $410,000 to provide initial funding for a project they are pursuing. They decide to do this by issuing 500 bonds. Each bond has a price of $820 and is repaid by the corporation with a single payment of $1000 at the end of 10 years. If the corporation sells all 500 bonds, then they will have raised the $410,000 they desired, and will settle their debt by paying a total of $500,000 in 10 years. In this scenario, the corporation will be paying an annual effective rate of interest that is slightly higher than 2%. The purchasers of the bonds will earn that same rate on their investments.

The scenario detailed above is an example of a **zero-coupon bond**. This means that the bond is settled with a single payment made by the borrower at some point in the future. Some bonds will make regular interest payments, called **coupons**, at regular intervals throughout the life of the loan. We will restrict our current discussion to zero-coupon bonds, but will discuss coupon-paying bonds in detail in a later section. When a bond is repaid by the borrower, it is said to have **matured**. The amount repaid to the investor is called the **maturity amount** of the bond.

Some short-term bonds (called T-bills) are priced using simple interest or simple discount formulas. We will study T-bills later in this section. Unless you are told specifically that you are working with a T-bill, you should assume compound interest is being used to price a bond. When working with such bonds, it is common to quote their rates using nominal annual rates compounded semiannually.

Example 1.50 A company issues a new zero-coupon bond. The bond is set to mature for 1000 in 15 years. The bond is priced to yield its purchaser a nominal annual rate of 3% convertible semiannually. Determine the price of the bond.

Example 1.51 A city government issues a 30-year zero-coupon bond. The bond matures for 1000 and has a price of 400. Determine the yield rate on the bond. Express your answer as a nominal annual rate of interest compounded semiannually.

Treasury Bills

The term **treasury bill**, or **T-bill**, refers to a short-term zero-coupon bond issued by the U.S. and Canadian treasuries. T-bills are available with terms of 4 weeks, 13 weeks, 26 weeks, or 52 weeks. T-bills differ from other types of zero-coupon bonds in that they are not priced using compound interest formulas. U.S. T-bills are priced using a variation of the simple discount formula, whereas Canadian T-bills are priced using the simple interest formula.

United States Treasury Bills

Quoted rates for U.S. Treasury bills are determined using the simple discount accumulation function, which is given by $a(t) = (1 - d\,t)^{-1}$. If P is the price of the bond, F is the maturity amount, and d is the quoted discount rate, then $F = P \cdot (1 - d\,t)^{-1}$ and $P = F \cdot (1 - d\,t)$.

There is one unusual aspect of U.S. T-bills that we need to be aware of. The pricing formula for U.S. Treasury bills assumes a 360-day year instead of a standard 365-day year. When working with (for example) a 4-week T-bill, one would use $t = 28 / 360$ in the pricing formula. For a 26-week T-bill, one would use $t = 182 / 360$.

United States Treasury Bills

- Assume a U.S. T-bill has a price of *P*, a maturity amount of *F*, and a quoted rate of *d*.
- Let *t* be the term of the bond in years, assuming a 360-day year.
- Then we have $P = F \cdot (1 - d\,t)$.
- We can also show that $d = \dfrac{360}{\text{Days to Maturity}} \cdot \dfrac{I}{F}$, where $I = F - P$.

Example 1.52 A new U.S. Treasury bill has a price of 990 and matures in 26 weeks for 1000. Find the quoted rate for the T-bill, as well as the annual effective rate of interest paid by the bond.

Example 1.53 A new U.S. Treasury bill matures in 13 weeks for 1000 and has a quoted rate of 2%. Find the price of the T-bill.

Canadian T-Bills

Canadian T-bills are priced using the simple interest accumulation function, and use a 365-day year. Formulas related to Canadian T-bills are provided below.

Canadian Treasury Bills

- Assume a Canadian T-bill has a price of *P*, a maturity amount of *F*, and a quoted rate of *i*.
- Let *t* be the term of the bond in years, assuming a 365-day year.
- Then we have $F = P \cdot (1 + i\,t)$.
- We can also show that $i = \dfrac{365}{\text{Days to Maturity}} \cdot \dfrac{I}{P}$, where $I = F - P$.

Example 1.54 A new Canadian Treasury bill has a price of 982 and matures in 52 weeks for 1000. Find the quoted rate for the T-bill, as well as the annual effective rate of interest paid by the bond.

Example 1.55 A new Canadian Treasury bill matures in 4 weeks for 1000 and has a quoted rate of 3%. Find the price of the T-bill.

1.8 FORCE OF INTEREST

Definition of Force of Interest

When working with compound interest, whether compounded continuously, annually, or on some non-annual basis, the effective annual interest rate is the same over any two periods of the same length. For general accumulation functions however, we have seen that the annual effective interest rate can (and likely will) vary when calculated over different periods of the same length. This fact is demonstrated in the next example.

Example 1.56 A fund grows according to the accumulation function $a(t) = 1+0.06t^2$. Find the annual effective interest rate during each quarter of the first year.

In the previous example, the rate at which interest is being accumulated isn't just changing from one quarter to the next, but is in fact changing from one instant to the next. In this section, we will introduce a mechanism for discussing continuously changing rates of growth. This tool will be called the force of interest.

Assume that we have an account whose value at time t is given by the amount function $A(t)$. We know from calculus that the derivative $A'(t)$ provides the instantaneous rate of growth in the value of the fund. The quantity $A'(t)$ provides a *absolute growth rate*, which is measured in units of currency per year. This is not a great tool for measuring a fund's performance, as it doesn't take into account the current value of the fund. For example, assume we know that an account has an instantaneous absolute growth rate of $A'(t) = 50$ units of currency per year at time *t*. That relative rate of growth is much more substantial if the current value of the account is $A(t) = 100$ than if the current value is $A(t) = 2000$.

Rather than an absolute growth rate, we would prefer to work with a *relative growth rate* that expresses the absolute growth rate as a proportion of the current value of the account. The expression $A'(t) / A(t)$ provides us with such a relative growth rate. We will refer to $A'(t) / A(t)$ as the **force of interest**, and will denote it by δ_t.

Let $a(t)$ be the accumulation function associated with the amount function $A(t)$. Then $A(t) = P \cdot a(t)$, where P is the initial principal in the fund. Noting that $A'(t) = P \cdot a'(t)$, we can show that $\delta_t = a'(t) / a(t)$ as follows:

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{P \cdot a'(t)}{P \cdot a(t)} = \frac{a'(t)}{a(t)}$$

Definition of Force of Interest

- Let $A(t)$ and $a(t)$ be the amount and accumulation functions for a fund, respectively.
- The **force of interest** of the fund at time t is defined to be $\delta_t = \dfrac{A'(t)}{A(t)} = \dfrac{a'(t)}{a(t)}$.

Note that, in general, the force of interest is a function of t and will change continuously over time.

Example 1.57 Find the force of interest δ_t for the accumulation function $a(t) = 1 + 0.06t^2$. Calculate δ_0, $\delta_{0.25}$, $\delta_{0.5}$, $\delta_{0.75}$, and δ_1.

Force of Interest as a Nominal Rate

The force of interest provides a means of measuring a continuously changing rate of growth associated with an accumulation function $a(t)$. The actual value of δ_t can be interpreted as a nominal annual rate of interest convertible continuously. To see that this is true, we will need to work with limits.

Let m be some positive number, and let $\Delta t = 1/m$. That is, Δt is a length of time equal to one mth of a year. Let $j_t^{(m)}$ equal the (non-annual) effective rate of interest during the period of time $[t,\ t + \Delta t]$. Then $j_t^{(m)}$ is equal to the amount of growth during this period divided by the value at the beginning of the period. That is:

$$j_t^{(m)} = \frac{a(t + \Delta t) - a(t)}{a(t)}$$

Now, let $i_t^{(m)}$ be the nominal annual rate convertible m-thly, calculated over the period $[t,\ t + \Delta t]$. Then we have:

$$i_t^{(m)} = j_t^{(m)} \cdot m = j_t^{(m)} \cdot \frac{1}{\Delta t} = \frac{a(t + \Delta t) - a(t)}{a(t)} \cdot \frac{1}{\Delta t} = \frac{a(t + \Delta t) - a(t)}{\Delta t} \cdot \frac{1}{a(t)}$$

Now define i_t^{∞} to be $i_t^{\infty} = \lim_{m\to\infty} i_t^{(m)}$. Note that as m approaches infinity, Δt approaches zero. We now have:

$$i_t^{\infty} = \lim_{m\to\infty}\left[\frac{a(t + \Delta t) - a(t)}{\Delta t} \cdot \frac{1}{a(t)}\right] = \lim_{\Delta t\to 0}\left[\frac{a(t + \Delta t) - a(t)}{\Delta t}\right] \cdot \frac{1}{a(t)} = a'(t) \cdot \frac{1}{a(t)} = \frac{a'(t)}{a(t)} = \delta_t$$

This verifies our earlier claim that δ_t is essentially a nominal annual rate of interest convertible continuously.

We will now consider additional examples related to the force of interest.

Example 1.58 Find the force of interest δ_t for each of the following accumulation functions.

a) $a(t) = 1 + 0.08t$
b) $a(t) = (1 + 0.01t)^2$
c) $a(t) = (1.05)^t$
d) $a(t) = e^{0.08t}$

Example 1.59 The value of an account at time t is given by the amount function $A(t) = 2000(1 + 0.05t)^2$.

a) Calculate the force of interest at the end of the first year.
b) Find the time at which the force of interest is equal to 0.08.

Example 1.60 The value of an account at time t is given by $A(t) = Pt^2 + Qt + R$. You are given that $A(0) = 200$, $A(1) = 232$, and $A(3) = 368$. Calculate the force of interest at time $t = 0.5$.

Finding the Accumulation Function for a Given Force of Interest

We have seen how to find the force of interest for a given accumulation function $a(t)$. However, most problems that you will encounter involving a force of interest will provide you with δ_t, from which you will need to reconstruct an accumulation function $a(t)$. To see how to do this, first observe that $\delta_t = a'(t)/a(t) = \frac{d}{dt}\ln[a(t)]$. Switching to a dummy variable r and then integrating from 0 to t gives us $\int_0^t \delta_r\,dr = \int_0^t \frac{d}{dr}\ln[a(r)]\,dr$. We can apply the fundamental theorem of calculus to the right hand side to obtain $\int_0^t \delta_r\,dr = \ln[a(t)]$. Exponentiating both sides then gives us the expression $a(t) = e^{\int_0^t \delta_r\,dr}$.

Finding an Accumulation Function Given a Force of Interest

- Given a force of interest δ_t, we can find the associated accumulation function using the formula $a(t) = e^{\int_0^t \delta_r\, dr}$.

Example 1.61

An account earns interest according to the force interest $\delta_t = \dfrac{0.2}{1 + 0.1t}$, for $t \geq 0$.

a) Find $a(t)$.
b) Assume 50 is deposited at time 0. Find the accumulated value at time 6.
c) Find the present value of a payment of 200 at time 5.
d) Find the present value of a payment of 200 at time 5 and a payment of 100 at time 10.
e) An amount of P is deposited at time 0. The account is worth 500 at time 5. How much is the account worth at time 8?

Example 1.62

An amount of P is deposited into each of two funds: Fund X and Fund Y.

- Fund X accumulates at a force of interest given by $\delta_t = \dfrac{t^2}{K}$.
- Fund Y earns a nominal annual rate of interest of 10% convertible semiannually

At the end of year 6, the accumulated values of the two funds are equal. Find K.

Finding Accumulation Functions Algebraically

Assume that you are provided with a force of interest that is written as a fraction where the numerator is equal to the derivative of the denominator. That is, assume that $\delta_t = f'(t) / f(t)$ for some function $f(t)$. Since δ_t is defined as $\delta_t = a'(t) / a(t)$, it would be tempting to assume that $f(t) = a(t)$ in this situation. That is not necessarily true, however. Assume that $a(t) = K \cdot f(t)$ for some constant K. Then $a'(t) = K \cdot f'(t)$, and thus $\delta_t = a'(t) / a(t) = f'(t) / f(t)$ even though $a(t) \neq f(t)$. It can, however, be shown that if $\delta_t = f'(t) / f(t)$, then $a(t) = K \cdot f(t)$ for some constant K. The actual value of K can be determined using the property that $a(0) = 1$.

Example 1.63

Find the accumulation functions associated with the following forces of interest.

a) $\delta_t = 0.04 / (1 + 0.04t)$
b) $\delta_t = 1 / (t + 8)$
c) $\delta_t = 0.1t / (1 + 0.05t^2)$
d) $\delta_t = t / (4 + 0.5t^2)$

Example 1.64

Phillip and Gary each deposit an amount P into separate funds. Phillip's fund earns a nominal annual rate of discount of 8% convertible quarterly. Gary's fund accumulates interest at a force of interest $\delta_t = 1 / (t + 10)$. After 6 years, Phillip's fund is worth 2500 and Gary's fund is worth K. Find K.

Example 1.65 An account earns interest according to the force of interest $\delta_t = 2k/(1+kt^2)$. The annual effective rate of interest earned by the account during year 6 is equal to 10%. Determine the force of interest at end end of year 6.

This algebraic method can, in theory, be used to find the accumulation function associated with any force of interest. However, as a result of simplification, the force of interest is often written in a way that is not immediately recognizable as being in the form $\delta_t = f'(t)/f(t)$. Consider the force of interest in Example 1.61. To use this algebraic method on this problem, once would have to recognize that $\delta_t = \dfrac{0.2}{1+0.1t} = \dfrac{0.2(1+0.1t)}{(1+0.1t)^2}$.

Force of Interest for Compound Interest

Consider the accumulation function $a(t) = (1+i)^t$, which is associated with compound interest with annual compounding. Calculating the force of interest for this accumulation function gives $\delta_t = \ln(1+i)$. Since i is a constant, the force of interest is constant for any compound interest accumulation function. For that reason, we will drop the subscript from δ_t when working with compound interest, simply stating that $\delta = \ln(1+i)$. It should be noted that the concept of force of interest for compound interest is exactly the same as the concept of a continuous rate of compound interest. If we are given a constant force of interest δ, our accumulation function could thus be written as $a(t) = e^{\delta t}$.

Example 1.66 Fund X accumulates interest at a force of interest of $\delta = r$. Fund Y earns simple interest at an annual rate of $i = r$. A deposit into Fund X will double in value over the course of 10 years. Determine how long it would take a deposit into Fund Y to double in value.

Example 1.67 The interest rate earned by an account varies each year, as follows:

- A nominal annual discount rate of 8% convertible quarterly is earned during year 1.
- A nominal annual interest rate of 6% convertible every 2 years is earned during year 2.
- A force of interest of 5% is earned during year 3.

Find the annual effective rate of interest during the three year period.

Accumulating over General Periods

As discussed in Section 1.3, if we are not working with compound interest, we would use the accumulation factor $a(t_1, t_2) = a(t_2)/a(t_1)$ to accumulate over a period from time t_1 to time t_2.

Example 1.68 An account accumulates at a force of interest $\delta_t = \dfrac{1}{1+t}$, $t \geq 0$. Assume 100 is deposited at time 3. Find the value of the account at time 5.

Example 1.69 A fund earns interest at a force of interest given by $\delta_t = \dfrac{0.06t}{1+0.03t^2}$. A deposit of 200 is made into the fund at time 0, and another deposit of 100 is made at at the end of year 5.

a) Find the value of the fund at the end of year 8.
b) Find the time t at which the value of the fund is equal to 1000.