CHAPTER 2 – Annuities

2.1 ANNUITY IMMEDIATE

An *annuity* is any series of periodically occurring payments. Annuities are an important concept and are frequently encountered in the fields of finance and economics. Before we begin our discussion of the mathematics related to annuities, we need to review the concept of a finite geometric series.

Finite Geometric Series

A **finite geometric sequence** is an ordered sequence of n numbers in which the ratio of any two consecutive numbers in the sequence is constant. If a is the first term in the sequence, and r is the common ratio, then the terms of the sequence would be $a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^{n-1}$.

A **finite geometric series** is the sum of a finite geometric sequence. A geometric series with initial term a, common ratio r, and with n total terms would have the form $a + ar + ar^2 + \ldots + ar^{n-1}$. A formula for the sum of a finite geometric series is given below.

Sum of a Finite Geometric Series

- Consider a **finite geometric series** of the form $a + ar + ar^2 + \ldots + ar^{n-1}$.
- The sum of this series is given by $a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a - ar^n}{1 - r} = \dfrac{a\left(1 - r^n\right)}{1 - r}$.
- A convenient way of remembering the formula for the sum of a finite geometric series is to read the $\dfrac{a - ar^n}{1 - r}$ version of the formula as $\dfrac{(\text{first term}) - (\text{first ommitted term})}{1 - (\text{common ratio})}$.

Example 2.1

Find the sum of the following finite geometric series.

a) $3 + 6 + 12 + 24 + 48$

b) $1 + (1.06) + (1.06)^2 + (1.06)^3 + \ldots + (1.06)^{10}$

Level Annuities

The payments in an annuity might be fixed at some level, might increase or decrease in some prescribed way, or might follow some other type of pattern. We will begin our study of annuities by considering those whose payments remain constant in size. Such annuities are called **level annuities**.

The payments in any annuity can be considered to be occurring either at the beginning of a time period, or at the end of a time period. If we are treating the payments as occurring at the end of a time period, then we call the annuity an **annuity immediate**. If, on the other hand, we consider the payments to be taking place at the beginning of a time period, we call the annuity an **annuity due**. We will first look at annuities immediate.

Annuity Immediate

Consider an annuity at makes payments of the end of each year for a total of n years. Since the payments occur at the end of the year, this annuity would be considered an **annuity immediate**. Note that the first payment would occur at time $t = 1$.

We begin by assuming that each of the payments is equal to 1. We wish to develop formulas for the present value of such an annuity at time $t = 0$, as well as its accumulated value at time $t = n$. Actuaries use special notation for the present value and accumulated value of annuity of this form.

- $a_{\overline{n}|i}$ denotes the PV at time $t = 0$ of an annuity that pays 1 at the end of each year for n years.
- $s_{\overline{n}|i}$ denotes the AV at time $t = n$ of an annuity that pays 1 at the end of each year for n years.

These symbols $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ are read as "*a* angle *n* i" and "*s* angle *n* i" respectively.

Using ideas from Chapter 1, we can show that $a_{\overline{n}|i} = v + v^2 + ... + v^n$ and $s_{\overline{n}|i} = 1 + (1+i) + (1+i)^2 ... + (1+i)^{n-1}$. Both of these series are finite geometric series, the first with common ratio v and the second with common ratio $(1+i)$. We can apply the formula for the sum of a finite geometric series, along with relationships between i and v to obtain the two formulas below, which are fundamental to the study of annuities:

$$a_{\overline{n}|i} = \frac{1 - v^n}{i} \quad \text{and} \quad s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

The annuity as a whole has the same time value as a single payment of $a_{\overline{n}|i}$ at time $t = 0$, and also the same time value as a single payment of $s_{\overline{n}|i}$ at time $t = n$. Such payments of $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ would thus have equal time values. It follows that the two values are related by the expressions $s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i}$, and $a_{\overline{n}|i} = v^n s_{\overline{n}|i}$.

If an annuity immediate has level annual payments of R instead of 1, then its present and accumulated values are greater by a factor of R, and are thus given by $R\,a_{\overline{n}|i}$ and $R\,s_{\overline{n}|i}$ respectively.

Annuities Immediate

- Consider an annuity immediate that makes payments of 1 at the end of each year.
- The present value of such an annuity at time $t = 0$ is denoted by $a_{\overline{n}|i}$ and calculated using:
  $a_{\overline{n}|i} = v + v^2 + ... + v^n = \frac{1 - v^n}{i}$
- The accumulated value of the annuity at time $t = n$ is denoted by $s_{\overline{n}|i}$ and calculated using:
  $s_{\overline{n}|i} = 1 + (1+i) + (1+i)^2 ... + (1+i)^{n-1} = \frac{(1+i)^n - 1}{i}$
- The values $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ are related by $s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i}$, and $a_{\overline{n}|i} = v^n s_{\overline{n}|i}$.
- The present and accumulated values of an annuity immediate that pays R at the end of each year are given by $R\,a_{\overline{n}|i}$ and $R\,s_{\overline{n}|i}$ respectively.

It is important to remember the following facts about the times associated with the values $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$:

- The value $a_{\overline{n}|i}$ gives the time value of the annuity **one year before** the first payment.
- The value $s_{\overline{n}|i}$ gives the time value of the annuity **at the time of** the last payment.

When only a single interest rate is being considered, we will often use the shorthand notations $a_{\overline{n|}}$ and $s_{\overline{n|}}$.

Example 2.2 Deposits of 75 are made into a fund at the end of each year for 24 years. The effective annual interest rate is 3%. Calculate the present value of this series of payments.

Example 2.3 Deposits of 120 are made into a fund at the end of each year for 8 years. The effective annual interest rate is 4.5%. Calculate the accumulated value of the series of payments at the end of the 8th year.

Example 2.4 Deposits of P are made into a fund at the end of each year for 12 years. At an effective annual interest rate of 6%, the accumulated value of the series of payments at the end of the 12th year is 2783.54. Find P.

Performing Annuity Calculations with the BA II Plus

Financial calculators such as the BA II Plus come equipped with "Time Value of Money (TVM) Calculators" for performing annuity calculations. Some brief examples of how to use the BA II Plus to perform annuity calculations are given below. I suggest reading the manual or watching YouTube videos to get full instructions.

1. To calculate $100\,a_{\overline{16|}5\%}$ you would enter:
   [2ND] [CLR TVM] 16 [N] 5 [I/Y] 100 [+/-] [PMT] [CPT] [PV]

2. To calculate $100\,s_{\overline{16|}5\%}$ you would enter:
   [2ND] [CLR TVM] 16 [N] 5 [I/Y] 100 [+/-] [PMT] [CPT] [FV]

3. Assume you know that $800 = 100\,a_{\overline{16|}i}$ and you wish to find i. You would enter:
   [2ND] [CLR TVM] 16 [N] 800 [PV] 100 [+/-] [PMT] [CPT] [I/Y]

Although the BA II Plus can be used to calculate $a_{\overline{n|}i}$ and $s_{\overline{n|}i}$, it is **strongly** recommended that you get used to performing such calculations using the TI-30X along with the formulas we have introduced. There are three reasons for this suggestion:

1. Once you get used to doing these calculations, you will almost certainly be able to perform them more quickly with the TI-30X than with the BA II Plus, and will likely make fewer errors.
2. It is necessary for you to learn the formulas $a_{\overline{n|}i} = \left(1 - v^n\right) / \, i$ and $s_{\overline{n|}i} = \left((1+i)^n - 1\right) / \, i$. The best way to learn these formulas is by using them.
3. Later in this chapter we will encounter more complicated types of annuities for which the BA II Plus can not easily be used to calculate their present or accumulated values. When working with such annuities, it will be helpful if you are comfortable with performing basic annuity calculations using the TI-30X.

That said, there are circumstances under which the BA II Plus is useful. Assume that you need to solve for the interest rate in an equation of the form $a_{\overline{n|}i} = K$. This would involve finding the roots of an nth degree polynomial. If n is greater than 3, then you will almost certainly require some form of technology to solve for i.

Solving for n in the equation $a_{\overline{n|}i} = K$ is certainly possible without the BA II Plus, but the process involves taking logarithms and it will likely be quicker to find n using the TVM calculator.

Example 2.5 Deposits of 60 are made into a fund at the end of each year for 16 years. The present value of the series of payments is 596.08. Find the effective annual interest rate.

Example 2.6

Amber and Bert each make deposits of 500 at the end of each year for 20 years. Amber's account earns an annual effective rate of 4% and Bert's account earns an annual effective rate of 6%.

After making deposits for 20 years, both people begin withdrawing money from their accounts. They each make withdrawals at the end of the year for 12 years, with the first withdrawal occurring exactly one year after the last deposit. Amber's withdrawals are in an amount of P each, while Bert's are each in an amount of Q.

a) Determine the value of each person's account at the end of 20 years.
b) Find P and Q.

Annuities with Non-Annual Payments

In our discussion of annuities, we have so far assumed that the payments were made annually. We will often encounter annuities where the payments are made every month, every 6 months, or quarterly. We can use the same formulas to calculate the PV and AV of these annuities. When dealing with non-annual payment periods, the n in the formulas for $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ should equal the total number of payment periods and i should be replaced with the effective periodic rate, j.

Example 2.7

Gary deposits P at the end of each month. His employer matches each deposit. His fund earns interest at a rate of 3% convertible monthly.

Gary makes deposits for 30 years, and then retires. After retirement, he withdraws $2000 at the end of each month for 20 years. After these 20 years, the account is empty.

Find P.

It is also possible to encounter annuities in which the payments occur less frequently than once a year, as seen in the next example.

Example 2.8

An annuity pays 50 every two years, with the first payment occurring at the end of year 1 and the last payment occurring at the end of year 29. Assume an annual effective interest rate of 5%.

a) Find the present value of this annuity.

b) Find the accumulated value of this annuity at the time of the last payment.

2.2 ANNUITY DUE

Annuity Due

As mentioned in the last section, we will occasionally wish to consider annuities in which the payments occur at the beginning of the year rather than at the end of the year. Such annuities are called **annuities due**. We introduce the following notation to refer to the present and accumulated values of an annuity due:

- $\ddot{a}_{\overline{n}|i}$ denotes the PV at time $t = 0$ of an annuity that pays 1 at the beginning of each year for n years.
- $\ddot{s}_{\overline{n}|i}$ denotes the AV at time $t = n$ of an annuity that pays 1 at the beginning of each year for n years.

We can use principles from Chapter 1 to show that $\ddot{a}_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i}$ can be written as $\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + ... + v^{n-1}$ and $\ddot{s}_{\overline{n}|i} = (1+i) + (1+i)^2 ... + (1+i)^n$. Applying formulas for the sum of a finite geometric sequence, as well as identities relating *i*, *v*, and *d*, we obtain the following formulas for $\ddot{a}_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i}$:

$$\ddot{a}_{\overline{n}|i} = \frac{1 - v^n}{d} \quad \text{and} \quad \ddot{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{d}$$

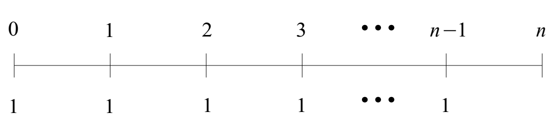
If the annual payments are in an amount of R rather than 1, then the present and accumulated values of the annuity due are given by $R\ddot{a}_{\overline{n}|i}$ and $R\ddot{s}_{\overline{n}|i}$ respectively.

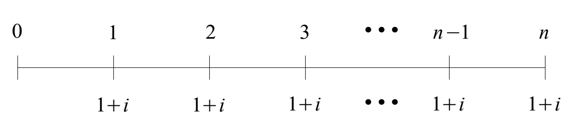
Notice that the formulas above are nearly identical to the formulas for $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$. The only difference is that the denominator is equal to i in the formulas for annuities immediate, and equal to d for annuities due. This presents a convenient mnemonic device: "Use *i* for annuities **(i)**mmediate, and use *d* for annuities **(d)**ue.**"**

While you should certainly memorize the formulas presented above for $\ddot{a}_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i}$, they are not generally the most efficient way to perform calculations relating to annuities due. These formulas require you to calculate the discount rate d. Although that is is not a difficult task, it does introduce a second rate that you have to keep track of. An alternate method of calculating $\ddot{a}_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i}$ involves appealing to relationships between annuities due and immediate. We now discuss one such relationship.

Annuity Immediate as Delayed Annuity Due

Notice that a payment of 1 at the beginning of a year has the same time value as a payment of $(1+i)$ at the end of the year. Given an annuity due paying 1 at the beginning of each year, we could postpone each payment, replacing them with payments of $(1+i)$ at the end of the year. In this way, we can convert an annuity due with payments of 1 into an annuity immediate with payments of $(1+i)$. These two annuities are illustrated in the time diagrams below. These annuities are equivalent, and will have the same time value at all times. This allows us to conclude that $\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}$.





Using the formulas $\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}$ to find $\ddot{a}_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i}$ saves us from needing to calculate the discount rate, *d*. These formulas also provide us with a method of using the BA-II Plus to calculate values such as $R\ddot{a}_{\overline{n}|i}$. Notice that $R\ddot{a}_{\overline{n}|i} = R(1+i)a_{\overline{n}|i}$. We can use the BA-II Plus to calculate the right-hand side of this equation by entering $R(1+i)$ as the payment. As similar approach can be used to calculate $R\ddot{s}_{\overline{n}|i}$.

We now summarize what we have learned about annuities due.

Annuities Due

- Consider an annuity due that makes payments of 1 at the beginning of each year for n years.
- The present value of such an annuity at time $t = 0$ is denoted by $\ddot{a}_{\overline{n}|i}$ and is equal to:

  $$\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + ... + v^{n-1} = \frac{1 - v^n}{d}$$

- The accumulated value of the annuity at time $t = n$ is denoted by $\ddot{s}_{\overline{n}|i}$ and is equal to:

  $$\ddot{s}_{\overline{n}|i} = (1+i) + (1+i)^2 ... + (1+i)^n = \frac{(1+i)^n - 1}{d}$$

- The time values of annuities due and annuities immediate are related by the following equations: $\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}$.
- The present and accumulated values of an annuity due that pays R at the beginning of each year are given by $R\ddot{a}_{\overline{n}|i}$ and $R\ddot{s}_{\overline{n}|i}$, respectively.

Example 2.9 Deposits of 45 are made into a fund at the beginning of each year for 18 years. The effective annual interest rate is 7.5%. Calculate the present value of this series of payments.

Example 2.10 Deposits of 80 are made into a fund at the beginning of each year for 10 years. The effective annual interest rate is 5%. Calculate the accumulated value of the series of payments at the end of the 10th year.

Example 2.11 Deposits of P are made into a fund at the beginning of each year for 15 years. At an effective annual interest rate is 4.5%, the present value of the series of payments is 729.48. Find P.

The following problem can be easily solved using the BA-II Plus and the relationship $\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}$.

Example 2.12 Deposits of 110 are made into a fund at the beginning of each year for T years. At an effective annual interest rate is 8%, the present value of the series of payments is 979.42. Find T.

Time of First and Last Payments

The end of any one year can be thought of as the beginning of the following year. This observations allows us to think of any annuity immediate as an annuity due, and vice versa. In some sense, the distinguishing characteristic between an annuity immediate and an annuity due is simply a matter of whether we are selecting the time $t=0$ to be the time of the first payment, or one year prior.

Since the difference between these two types of annuities is primarily a matter of perspective, it would be useful to discuss the different results yielded by using one type of annuity over the other. These are summarized below.

- Annuities Immediate
  - $a_{\overline{n}|i}$ gives the present value of the annuity one period **BEFORE** the first payment.
  - $s_{\overline{n}|i}$ gives the accumulated value of the annuity at the **SAME TIME** as the last payment
- Annuities Due
  - $\ddot{a}_{\overline{n}|i}$ gives the present value of the annuity at the **SAME TIME** as the first payment.
  - $\ddot{s}_{\overline{n}|i}$ gives the accumulated value of the annuity one period **AFTER** the last payment.

It is important to remember these rules. Doing so will allow you a degree of flexibility in working with annuity problems. There are situations were it is more convenient to treat an annuity as an annuity due than as an annuity immediate, and vice versa.

Example 2.13 Deposits of 25 are made into a fund at the end of each year for 8 years with the first deposit occurring at t = 4. The effective annual interest rate is 6%. Calculate the present value of the series of payments.

Example 2.14 Deposits of 40 are made into a fund at the beginning of each year with the first deposit occurring at t = 8. The effective annual interest rate is 5%. Calculate the accumulated value of the series of payments at the end of the 26th year.

The "Plus One / Minus One" Formulas

We will now discuss a second relationship that exists between the formulas for annuities immediate and due. These new formulas are informally called the **Plus One / Minus One Formulas**. We will first state these formulas, and then discuss why they are true and when they should be used.

Plus One / Minus One Formulas

The following equations hold for all values of n and i

- $\ddot{a}_{\overline{n}|i} = a_{\overline{n-1}|i} + 1$
- $\ddot{s}_{\overline{n}|i} = s_{\overline{n+1}|i} - 1$

To see that $\ddot{a}_{\overline{n}|i} = a_{\overline{n-1}|i} + 1$ is true, note that $\ddot{a}_{\overline{n}|i}$ gives the present value of a sequence of n annual payments of 1, with the first payment occurring at time $t=0$. Imagine that we temporarily remove the payment at $t=0$. The remaining payments will occur at times $1, 2, 3, \ldots, n-1$, and can thus be thought of as an $(n-1)$-year annuity immediate. The present value of this annuity at time $t=0$ is given by $a_{\overline{n-1}|i}$. If we add back in the present value of the first payment, which is already at time $t=0$, then we get that the total present value for the n payments is equal to $a_{\overline{n-1}|i} + 1$. We thus conclude that $\ddot{a}_{\overline{n}|i} = a_{\overline{n-1}|i} + 1$.

A similar approach can be used to show that $\ddot{s}_{\overline{n}|i} = s_{\overline{n+1}|i} - 1$. Consider an annuity making n payments of 1 at times $0, 1, 2, \ldots, n-1$. The accumulated value of this annuity at time $t=n$, one year after the last payment, is $\ddot{s}_{\overline{n}|i}$. Now add another payment of 1 at time $t=n$. The total accumulated value of this sequence of n+1 payments at time $t=n$ is equal to $\ddot{s}_{\overline{n}|i} + 1$. But this new sequence forms an $(n+1)$-year annuity, which we are valuing at the time of the last payment. Thus the time $t=n$ accumulated value is also equal to $s_{\overline{n+1}|i}$. This tells us that $\ddot{s}_{\overline{n}|i} + 1 = s_{\overline{n+1}|i}$, and thus $\ddot{s}_{\overline{n}|i} = s_{\overline{n+1}|i} - 1$.

These formulas are particular useful if we need to solve for the interest rate in a problem involving an annuity due. To solve for a rate, we need to use the TVM calculator in the BA-II Plus. However, since the rate is unknown, the formulas $\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}$ cannot help us in this scenario.

Example 2.15 An annuity makes annual payments of 2 at the beginning of each year for 10 years. The present value of the annuity is 16. Find the annual effective rate of interest.

Example 2.16 Assume an annual effective interest rate of i. At this rate, the present value of an n-year annuity immediate with annual payments of 500 is equal to 4559.29. At the same rate, the present value of an $(n-1)$-year annuity immediate with annual payments of 400 is equal to 3411.57. Find i.

2.3 PERPETUITIES

A perpetuity is an annuity that makes periodic payments forever. As with standard annuities, we will consider two types of perpetuities: perpetuities immediate that make payments at the end of each year, and perpetuities due that make payments at the beginning of each year.

Perpetuity Immediate

A perpetuity immediate is a perpetuity in which the first payment occurs one year after the creation of the perpetuity, with payments continuing annually forever. Equivalently, we can say that payments occur at the end of the year. The present value of a perpetuity immediate is denoted by $a_{\overline{\infty}|i}$, and is given by the infinite geometric series $a_{\overline{\infty}|i} = v + v^2 + v^3 + \ldots$. Using the formula for the sum of an infinite geometric series, we see that $a_{\overline{\infty}|i} = 1 / i$.

Example 2.17 An alumnus of State University wants to make a donation to fund an annual scholarship. The alumnus will deposit *P* into a fund earning a 4% annual effective rate of interest. The account will be used to fund annual scholarships of 2000, with the first scholarship to be awarded one year after the deposit. The scholarships are intended to last forever. Find *P*.

The payments in an annuity are equal to the interest earned by the account each year. Since the payments are exactly equal to the interest earned, the balance of the account is the same after each payment, and is equal to the initial principle. Thus, the payments will always be in the same amount, and since they do not decrease the amount of principal invested, they will last forever.

Example 2.18 At an annual effective interest rate of *i*, a 20-year annuity immediate with annual payments of 1429 has the same present value as that of a perpetuity immediate with annual payments of 1000. Find *i*.

Example 2.19 Jackie purchases a perpetuity immediate that pays 50 at the end of each year forever. Jackie pays *P* for the perpetuity, which would earn her an annual effective interest rate of 8% on her purchase.

Five years after purchasing the annuity, immediately after receiving the fifth payment, Jackie sells the perpetuity to Frankie for a price of *Q*. Taking into account her sale of the annuity, Jackie ultimately earned an annual effective rate of 7% on her original investment of *P*.

Determine the annual effective yield rate that Frankie would see on his investment of *Q*.

Perpetuity Due

A perpetuity due is a perpetuity in which the first payment occurs at the time of the creation of the perpetuity, with payments continuing annually forever. Equivalently, we can say that payments occur at the beginning of the year. The present value of a perpetuity due is denoted by $\ddot{a}_{\overline{\infty}|i}$, and is given by $\ddot{a}_{\overline{\infty}|i} = 1 + v + v^2 + v^3 + \ldots$. Summing this infinite geometric series yields the formula $\ddot{a}_{\overline{\infty}|i} = 1 / d$. As with annuities, it is generally easier to calculate the present value of a perpetuity due by exploiting relationships that exist between perpetuities immediate and due.

Relationships Between Perpetuity Immediate and Perpetuity Due

As with standard annuities, it is true that $\ddot{a}_{\overline{\infty}|i} = (1+i)a_{\overline{\infty}|i}$. This can be seen by noting that payments in a perpetuity immediate lag one year behind those those in a perpetuity due. This identity can also be established algebraically as follows: $\ddot{a}_{\overline{\infty}|i} = 1/d = (1+i)/i = (1+i)a_{\overline{\infty}|i}$.

Another useful relationship between the present value formulas for perpetuities immediate and due follows from the the facts that $a_{\overline{\infty}|i} = v + v^2 + v^3 + \ldots$ and $\ddot{a}_{\overline{\infty}|i} = 1 + v + v^2 + v^3 + \ldots$. We can see from these definitions that $\ddot{a}_{\overline{\infty}|i} - a_{\overline{\infty}|i} = 1$, and thus that $\ddot{a}_{\overline{\infty}|i} = 1 + a_{\overline{\infty}|i}$.

We now summarize what we have learned about perpetuities.

Perpetuities

- The present value of a **perpetuity immediate** that pays 1 at the end of each year, continuing forever, is given by $a_{\overline{\infty}|i} = v + v^2 + v^3 + \ldots = \dfrac{1}{i}$.
- The present value of a **perpetuity due** that pays 1 at the beginning of each year, continuing forever, is given by $\ddot{a}_{\overline{\infty}|i} = 1 + v + v^2 + \ldots = \dfrac{1}{d}$.
- The following identities can be derived from the definitions of perpetuities immediate and perpetuities due.
  - $\ddot{a}_{\overline{\infty}|i} = (1+i)a_{\overline{\infty}|i}$
  - $\ddot{a}_{\overline{\infty}|i} = 1 + a_{\overline{\infty}|i}$

Example 2.20

The following annuities all have the same present value.

a) A perpetuity due with annual payments of 100, at an annual effective interest rate of i.

b) A perpetuity immediate with annual payments of 135, at an annual effective interest rate of $1.25i$.

c) An n-year annuity immediate with annual payments of 148, at an annual effective interest rate of i.

Find n.

Non-Annual Payments

The formulas above can be applied to perpetuities with non-annual payments. As with annuities, we would simply replace the annual effective rate i with the effective periodic rate associated with the payment period.

Example 2.21

A perpetuity pays 2 at the end of each odd-numbered year, and 5 at the end of each even-numbered year. Find the present value of this perpetuity at $i = 10\%$.

Example 2.22

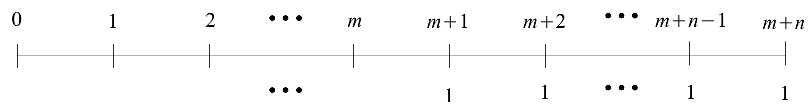
A perpetuity-immediate makes payments in the following sequence, forever: 1, 2, 3, 1, 2, 3, … . Find the present value of this perpetuity at $i = 10\%$.

The present value of the perpetuity in Example 2.22 can also be found using a technique called fusion. We will discuss this method in Section 2.5.

2.4 DEFERRED ANNUITIES AND BLOCK PAYMENTS

Deferred Annuities

You will occasionally encounter an annuity in which the first payment period does not begin at time $t = 0$. Such an annuity is called a **deferred annuity**. We will use the symbol ${}_{m|}a_{\overline{n}|i}$ to refer to the time $t = 0$ present value of a deferred annuity immediate that makes n annual payments of 1, with the first payment period beginning at time $t = m$. The first payment of this annuity would occur at time $t = m + 1$, and the last payment would occur at time $t = m + n$. A time diagram for such an annuity is pictured below.



The are two commonly used approaches for calculating the present value ${}_{m|}a_{\overline{n}|i}$:

1. Calculate the present value of the n payments at time $t = m$. This present value would be equal to $a_{\overline{n}|i}$. We then discount this quantity by m years to get the present value at time $t = 0$. This results in the formula ${}_{m|}a_{\overline{n}|i} = v^m a_{\overline{n}|i}$.

2. Start by considering an annuity immediate that makes payments for $m + n$ years. The present value of this annuity would be $a_{\overline{m+n}|i}$. We then subtract from this quantity the present value of the first n "missing" payments. This provides us with the formula ${}_{m|}a_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{n}|i}$.

The second method mentioned above is a special case of the block payments method of calculating present value. We will discuss this method later in this section.

Deferred Annuities

The symbol ${}_{m|}a_{\overline{n}|i}$ represents the present value of an n-year annuity immediate that pays 1 at the end of each year, with the first payment period starting at time $t = m$. The first payment of this annuity would occur at time $t = m + 1$ and the last payment would occur at time $t = m + n$. We present two formulas for calculating ${}_{m|}a_{\overline{n}|i}$:

- ${}_{m|}a_{\overline{n}|i} = v^m a_{\overline{n}|i}$
- ${}_{m|}a_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{n}|i}$

Example 2.23 Deposits of 60 are made into a fund at the beginning of each year for 8 years with the first deposit occurring at time $t = 10$. The effective annual interest rate is 4%. Calculate the present value of this series of payments.

Example 2.24 A company is planning for two sequences of future payments that they are obligated to deliver.

- The first sequence consists of semi-annual payments of 500 lasting for 6 years, with the first payment taking place exactly 5 years from today.
- The second sequence consists of semi-annual payments of 750 lasting for 6 years, with the first payment taking place exactly 8 years from today.

To cover these payments, the company will make semi-annual deposits of K into an account earning a nominal annual rate of 6%, convertible semi-annually. The deposits will last for 5 years, with the first deposit taking place today. Find K.

Block Payments

The method of **block payments** provides a convenient tool for calculating the present and accumulated values of an annuity in which the payments vary over time, but are constant for certain periods of time. An example of this would be a 10-year annuity immediate that made annual payments of 100 for the first 4 years, payments of 150 for the next 3 years, and payments of 300 for the final 3 years. The formulas for working with block payments are described below.

Block Payments

- Let $t_1, t_2, \ldots, t_n$ be an increasing sequence of whole numbers.
- Consider an annuity immediate that makes annual payments as follows:
  - Payments of P_1 are made at the end of years 1 through t_1.
  - Payments of P_2 are made at the end of years $t_1 + 1$ through t_2.

  ...
  - Payments of P_n are made at the end of years $t_{n-1} + 1$ through t_n.
- For $i = 1, 2, 3, \ldots, n-1$, let $\Delta_i = P_{i+1} - P_i$.
- The present value of this annuity at time $t = 0$ is given by:
  $PV = P_n a_{\overline{t_n}|} - \Delta_{n-1} a_{\overline{t_{n-1}}|} - \Delta_{n-2} a_{\overline{t_{n-2}}|} - \ldots - \Delta_2 a_{\overline{t_2}|} - \Delta_1 a_{\overline{t_1}|}$
- The accumulated value of this annuity at time $t = t_n$ is given by:
  $AV = P_1 s_{\overline{t_n}|} + \Delta_1 a_{\overline{t_n - t_1}|} + \Delta_2 a_{\overline{t_n - t_2}|} + \ldots + \Delta_{n-1} a_{\overline{t_n - t_{n-1}}|}$

The formulas presented above for block payments are complicated and likely somewhat confusing without additional context. One should probably not spend time attempting to memorize these formulas. The idea underlying block payments is probably best explained through examples.

Example 2.25 An annuity immediate pays 2 during years 1 – 6, and pays 8 during years 7 – 10. If $i = 10\%$, find the present value of this annuity, as well as the accumulated value at the end of year 10.

Example 2.26 An annuity immediate pays 5 during years 1 – 2, pays 3 during years 3 – 4, pays 9 during years 5 – 6, and pays 7 during years 7 – 8. Assume an annual effective interest rate of 7%.

a) Find the present value of this annuity at time $t = 0$.

b) Find the accumulated value of this annuity at time $t = 8$.

Example 2.27 Deposits are made into an account at the end of each year for $3n$ years as follows:

- Deposits of 75 are made for the first n years.
- Deposits of 100 are made for the middle n years.
- Deposits of 125 are made for the final n years.

The accumulated value of the account at the end of $3n$ years is equal to 9609. The annual effective interest rate earned by the account is i. You are given that $(1+i)^n = 2$. Find i.

Example 2.28 Kevin borrows 10,000 at an annual effective interest rate of 6% and agrees to repay it by making 30 annual payments with the first payment due in one year. The size of the payments is set to double after the first ten years. After making the tenth payment, Kevin is given the option of repaying the loan by making a final payment of K. This would result in Kevin paying an annual effective rate of 7% over the lifetime of the loan. Find K.

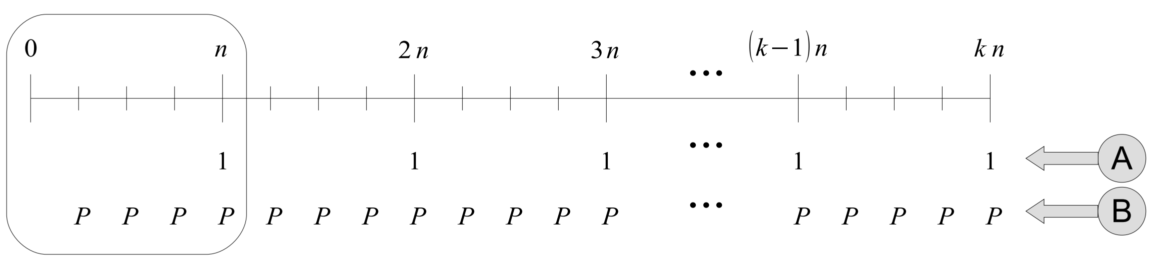
2.5 FUSION AND FISSION

There are times when it is convenient to convert a given annuity to one with equal present value, but with payments occurring either more or less frequently than the original annuity. The methods of Fission and Fusion allow us to "combine" or "split" payments in an annuity in order to achieve this goal.

Fission

The **fission** method is used to split annuity payments, creating a new annuity with more frequent payments. The process works as follows:

- Assume Annuity A makes k payments of 1 with payments occurring at the end of each n-year period. This is illustrated in the time diagram below with $n = 4$.
- We want to find an equivalent annuity (i.e. one with the same PV) that makes payments of P at the end of each year for the entire kn year period. Our goal is to find the appropriate value for P. We will call this new annuity Annuity B. It is also shown in the time diagram below.
- Notice that the first n payments in Annuity B must be equivalent to the payment of 1 at time n in Annuity A. It follows that $1 = P s_{\overline{n}|i}$ and $P = 1 / s_{\overline{n}|i}$.
- The PV of either annuity is thus $PV = P a_{\overline{kn}|i} = \dfrac{a_{\overline{kn}|i}}{s_{\overline{n}|i}}$.
- We conclude that the PV of an annuity paying 1 at the end of each n-year period for kn years is $\dfrac{a_{\overline{kn}|i}}{s_{\overline{n}|i}}$.



Fission

- The PV of an annuity paying 1 at the end of each n-year period for kn years is $\dfrac{a_{\overline{kn}|i}}{s_{\overline{n}|i}}$.

We could employ a similar strategy to show that the PV of an annuity paying 1 at the beginning of each n-year period for kn years is $\dfrac{a_{\overline{kn}|i}}{a_{\overline{n}|i}}$.

The discussion above explains how to convert an annuity that makes payments less frequently than annually into an annuity that pays annually. The same process could be used to convert any annuity into one that makes payments more frequently, regardless of what the actual periods are. For instance, you could use fission to convert an annual annuity into a monthly annuity.

You should be familiar with the the formula that we have derived above, as well as the process used to obtain it.

Converting Rates

There are many problems for which fission is a valid strategy, but that can be solved more easily by simply converting rates. For instance, assume you want to find the present value of an annuity that pays R every 3 years for 30 years. If you know the effective annual interest rate i, then you can find the effective 3 year rate j with $(1+i)^3 = 1+j$. The present value of the annuity could then be calculated using $PV = R\,a_{\overline{10}|j}$. The fission method is most useful when the interest rate is unknown, and thus cannot be converted. It is also useful for certain types of symbolic problems.

Example 2.29

An annuity pays 10 at the end of each 4 year period for 28 years. The effective annual rate of interest is $i = 6\%$.

a) Find the PV of this annuity by converting rates.
b) Find the PV of this annuity using fission.

Example 2.30

You are given the following information.

i. The present value of an n-year annuity immediate with payments of 1 at the end of every year is 16.4978.
ii. The present value of an n-year annuity immediate with payments of 1 at the end of every two years is 8.0322.
iii. The present value of an n-year annuity immediate with payments of 1 at the end of every three years is X.

Find X.

Example 2.31

Assume an annual effective interest rate of i. You are given the following:

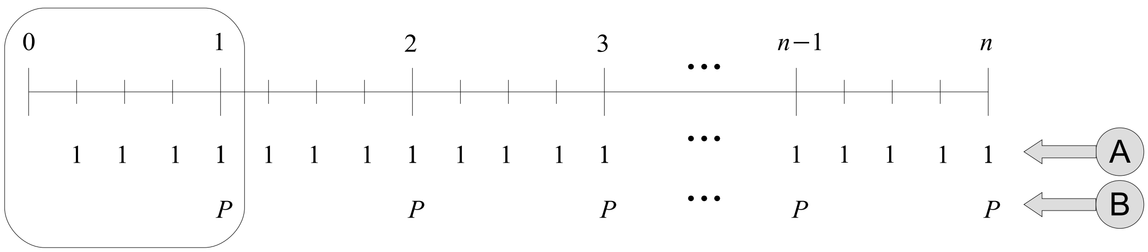
i. The present value of an annuity immediate paying 1 every n years for $4n$ years is equal to 2.6182.
ii. the present value of an annuity immediate paying 1 every $2n$ years for $4n$ years is equal to 1.1933.

Find the present value of an annuity immediate paying 1 at the end of each year for $4n$ years.

Fusion

Fusion is a method of combining annuity payments to create a new annuity with less frequent payments than the original one. A description of the method is provided below.

- Assume Annuity A makes m-thly payments of 1 for n years with payments occurring at the end of each $1/m$ year period. This is illustrated in the time diagram below with $m = 4$.
- We wish to find a second annuity, which we will call Annuity B, that makes payments of P at the end of each year for the entire n year period. Our goal is to find the appropriate value for P.
- Notice that the first m payments of 1 in Annuity A must be equivalent to the payment of P at time 1 in Annuity B. It follows that $P = s_{\overline{m}|j}$, where j is the effective m-thly rate.
- The PV of either annuity is thus $PV = P a_{\overline{n}|i} = \left(s_{\overline{m}|j}\right)\left(a_{\overline{n}|i}\right)$.



> **Fusion**
>
> - The PV of an annuity paying 1 at the end of each $(1/m)$-year period for n years is given by $PV = P a_{\overline{n}|i} = \left(s_{\overline{m}|j}\right)\left(a_{\overline{n}|i}\right)$ where j is the effective m-thly rate and i is the annual effective rate.

Since we are required to know j to use the fusion method, it is often unnecessary to use fusion to solve this sort of problem. If we already know j, then we could have calculated the present value of Annuity A by using the formula $PV = a_{\overline{nm}|j}$.

Fusion is most useful when the payments in the annuity vary in some sort of periodic manner. It is also common to see fusion employed in symbolic problems.

Example 2.32

You are given a perpetuity with annual payments as follows:

i) Payments of 2 at the end of the first year, and every three years thereafter.
ii) Payments of 7 at the end of the second year, and every three years thereafter.
iii) Payments of 4 at the end of the third year, and every three years thereafter.

The interest rate is 8% annual effective. Find the present value of this perpetuity.

Example 2.33

An annuity immediate makes n payments per year for 5 years. The size of the individual payments is equal to P during the first year, $2P$ during the second year, $3P$ during the third year, $4P$ during the fourth year, and $5P$ during the fifth year. The present value of the first n payments of P is equal to 140. Assuming an annual effective interest rate of 6%, find the present value of this annuity.

m-thly Paying Annuities

As an alternative to using fusion or converting rates, you may use the formulas detailed below to calculate the present value or accumulated value of an annuity that makes m-thly payments. It doesn't hurt to memorize these formulas, but it should not be a priority. Most problems of this type can be solved just as easily using other techniques you are familiar with. You should understand the notation used in these formulas, however.

- When an annuity symbol includes a superscript of (m), this indicates that the annuity pays out a total of 1 over the course of each year, but does so in m-thly installments of $1/m$.
- Assume an n-year annuity immediate makes m-thly payments of $1/m$.
  - The PV of this annuity is $a_{\overline{n}|}^{(m)} = \dfrac{1-v^n}{i^{(m)}}$.
  - The AV of this annuity is $s_{\overline{n}|}^{(m)} = \dfrac{(1+i)^n - 1}{i^{(m)}}$.
- Assume an n-year annuity due makes m-thly payments of $1/m$.
  - The PV of this annuity is $\ddot{a}_{\overline{n}|}^{(m)} = \dfrac{1-v^n}{d^{(m)}}$.
  - The AV of this annuity is $\ddot{s}_{\overline{n}|}^{(m)} = \dfrac{(1+i)^n - 1}{d^{(m)}}$.
- Assume a perpetuity immediate makes m-thly payments of $1/m$.
  - The PV of this perpetuity is $a_{\overline{\infty}|}^{(m)} = \dfrac{1}{i^{(m)}}$.
- Assume a perpetuity due makes m-thly payments of $1/m$.
  - The PV of this perpetuity is $\ddot{a}_{\overline{\infty}|}^{(m)} = \dfrac{1}{d^{(m)}}$.
- When dealing with an expression such as $R\,a_{\overline{n}|}^{(m)}$, it is important to remember that R represents the *total* of the payments made over the course of the year, and not the individual payments themselves.

Example 2.34

Which of the following statements are true?

(I) $s_{\overline{n}|} - a_{\overline{n}|} = i\,a_{\overline{n}|}s_{\overline{n}|}$

(II) $\ddot{s}_{\overline{n}|}^{(m)} - s_{\overline{n}|}^{(m)} = \dfrac{i^{(m)}}{m}\,s_{\overline{n}|}^{(m)}$

(III) ${}_{1/4|}\ddot{a}_{\overline{n}|}^{(2)} + a_{\overline{n}|}^{(2)} = a_{\overline{n}|}^{(4)}$

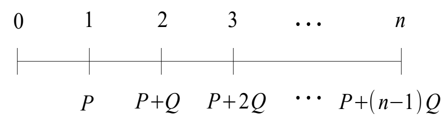
(A) I only (B) III only (C) I and II only
(D) II and III only (E) I , II, and III

2.6 ARITHMETIC ANNUITIES

Up to this point, we have primarily worked with level annuities in which the payments stay constant from one period to the next. In this section and the next, we will consider annuities in which the payments increase or decrease in some prescribed manner. We will consider annuities where the payments change arithmetically in this section, and in the next section we will consider annuities where the payments form a geometric sequence.

General Arithmetic Annuities (P/Q Formulas)

Consider an n-year annuity immediate in which the first payment is equal to P and each subsequent payment increases by a fixed amount Q. This is the general form for an arithmetic annuity. A time diagram for such an annuity is shown on the right.



Let A denote the present value of this annuity. We will now derive a formula for A.

1. Notice that $A = Pv + (P+Q)v^2 + (P+2Q)v^3 ... + (P+(n-1)Q)v^n$.
2. This expression can be rewritten as $A = P\left[v + v^2 + ... + v^n\right] + Q\left[v^2 + 2v^3 + ... + (n-1)v^n\right]$.
3. Since $a_{\overline{n}|} = v + v^2 + ... + v^n$, we see that $A = P a_{\overline{n}|} + Q\left[v^2 + 2v^3 + ... + (n-1)v^n\right]$.
4. Let $X = v^2 + 2v^3 + ... + (n-1)v^n$. Then $A = P a_{\overline{n}|} + QX$.
5. Notice that $(1+i)X = v + 2v^2 + ... + (n-1)v^{n-1}$.
6. It follows that $(1+i)X - X = v + v^2 + v^3 + ... + v^{n-1} - (n-1)v^n$.
7. This simplifies to $iX = v + v^2 + v^3 + ... + v^{n-1} + v^n - nv^n$.
8. If follows that $iX = a_{\overline{n}|} - nv^n$, and $X = \dfrac{a_{\overline{n}|} - nv^n}{i}$.
9. Thus, we see that $A = P a_{\overline{n}|} + Q\dfrac{a_{\overline{n}|} - nv^n}{i}$.

Let S be the accumulated value of this arithmetic annuity at time $t = n$. A formula for S can be obtained using a method similar to that which we used above to find the formula for A. Alternately, we could multiply the expression we derived for A by $(1+i)^n$. Doing so would yield the formula $S = P s_{\overline{n}|} + Q\left(s_{\overline{n}|} - n\right)/i$.

Arithmetic Annuities (*P*/*Q* Formulas)

Consider an n-year annuity immediate in which the first payment is equal to P , and each subsequent payment increases by a fixed amount Q .

- The present value of this annuity at time $t = 0$ is given by $A = P a_{\overline{n}|} + Q\dfrac{a_{\overline{n}|} - nv^n}{i}$.
- The accumulated value of this annuity at time $t = n$ is given by $S = P s_{\overline{n}|} + Q\dfrac{s_{\overline{n}|} - n}{i}$.

Example 2.35

At i = 8%, find the present value and accumulated value at $t = 6$ for the 6-year annuities immediate whose payments are given by each of the following sequences:

a) 12, 14, 16, 18, 20, 22

b) 20, 17, 14, 11, 8, 5, 2

Calculator Tip: An expression such as $S = 12 s_{\overline{6|}8\%} + 2\dfrac{s_{\overline{6|}8\%} - 6}{0.08}$ can be quickly calculated using the TI-30X by entering the following two commands: ▶ $\dfrac{1.08^{6} - 1}{0.08} \rightarrow x$ ▶ $12 * x + 2 * \dfrac{x - 6}{0.08}$

Example 2.36

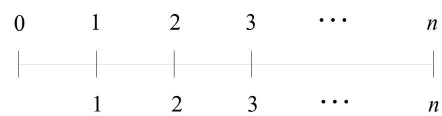
Assuming an annual effective interest rate of 7%, the following annuities have the same PV:

- An annuity immediate making quarterly payments of R for 10 years.
- An increasing annuity immediate with 10 annual payments, with the first payment in the amount of 400, and with subsequent payments increasing by 50 each year.

Find R.

Standard Increasing Annuities

Consider the special case of an arithmetic annuity in which $P = Q = 1$. This is an increasing annuity in which the payment at the end of any given year is equal to the number of years that have passed. A time diagram for such an annuity is shown on the right.



Annuities such as this are encountered frequently enough that we will introduce special notation for working with them. We will denote the present value of such an annuity by $(Ia)_{\overline{n|}}$ and will let $(Is)_{\overline{n|}}$ refer to the accumulated value of the annuity. By making the substitutions $P = Q = 1$ into the general formulas for arithmetic annuities and then simplifying, we obtain the following formulas.

Standard Increasing Annuities

Consider an n-year annuity immediate whose payments follow the sequence 1, 2, 3, …, n.

- The present value of this annuity at time $t = 0$ is given by $(Ia)_{\overline{n|}} = \dfrac{\ddot{a}_{\overline{n|}} - n v^{n}}{i}$.
- The accumulated value of this annuity at time $t = n$ is given by $(Is)_{\overline{n|}} = \dfrac{\ddot{s}_{\overline{n|}} - n}{i}$.

Notice that if an annuity immediate makes payments following the sequence R, $2R$, $3R$, …, nR, then its present and accumulated values are given by $R(Ia)_{\overline{n|}}$ and $R(Is)_{\overline{n|}}$, respectively.

Example 2.37

At $i = 7\%$, find the present value of an 12-year annuity immediate whose payments are given by the following sequence: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48.

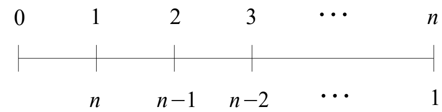
Calculator Tip: The value of an expression such as $4(Ia)_{\overline{12|}7\%}$ can be quickly calculated using the TI-30X by entering the following two commands: ▶ $\dfrac{1 - 1.07^{-12}}{0.07} \rightarrow x$ ▶ $4 * \dfrac{1.07 * x - 12 * 1.07^{-12}}{0.07}$

Example 2.38

Julia makes deposits at the end of each year into an account earning an annual effective interest rate of i. The first deposit is equal to 200, and subsequent deposits increase by 200 each year. The amount of interest earned by Julia's account during the tenth year is equal to 800. Find i.

Standard Decreasing Annuities

We now consider the special case of an arithmetic annuity in which $P = n$ and $Q = -1$. This results in a decreasing n-year annuity that pays n at $t=1$, with payments decreasing by 1 each year until reaching a final payment of 1 at $t=n$. A time diagram for such an annuity is shown on the right.



We will denote the present value of such an annuity with $(Da)_{\overline{n}|}$ and will denote the accumulated value by $(Ds)_{\overline{n}|}$. We obtain formulas for these values by substituting $P = n$ and $Q = -1$ into the general formulas for arithmetic annuities and then simplifying.

Standard Decreasing Annuities

Consider an n-year annuity immediate whose payments follow the sequence $n, n-1, \ldots, 3, 2, 1$.

- The present value of this annuity at time $t = 0$ is given by $(Da)_{\overline{n}|} = \dfrac{n - a_{\overline{n}|}}{i}$.
- The accumulated value of this annuity at time $t = n$ is given by $(Ds)_{\overline{n}|} = \dfrac{n(1+i)^n - s_{\overline{n}|}}{i}$.

If an annuity immediate makes payments following the sequence nR, $(n\text{-}1)R$, …, $3R$, $2R$, R, then its present and accumulated values are given by $R(Da)_{\overline{n}|}$ and $R(Ds)_{\overline{n}|}$, respectively.

Example 2.39 At $i = 6\%$, find the present value of a 15-year annuity immediate whose payments are given by the following sequence: 45, 42, 39, 36, 33, 30, 27, 24, 21, 18, 15, 12, 9, 6, 3.

Calculator Tip: The value of an expression such as $3(Da)_{\overline{15}|6\%}$ can be quickly calculated using the TI-30X by entering the following two commands: ▶ $\dfrac{1 - 1.06^{-15}}{0.06} \to x$ ▶ $3 * \dfrac{15 - x}{0.06}$

Example 2.40 At an annual effective interest rate of 8%, the following annuities have the same present value:

- A 12-year annuity immediate that pays $50t$ at the end of year t.
- A 12-year annuity immediate that pays $P(13 - t)$ at the end of year t.

Find P.

Non-Annual Annuities with Annual Increases

Assume you have an annuity that makes m-thly payments for $m > 1$. Further assume that the payments are K during the entire first year, but increase by an amount of D at the end of each year, to remain constant for another full year. The present or accumulated value of such an annuity can be calculated using fusion. We fuse together the first years worth of payments to obtain an initial annual fused payment of $P = K\, s_{\overline{m}|j}$. We then fuse together the increases in the payments to obtain an annual fused increase of $Q = D\, s_{\overline{m}|j}$. We can then apply the P/Q formulas to calculate the present and accumulated values of this annuity.

Example 2.41 An 8-year annuity immediate makes quarterly payments. The payments are 5 per quarter during the first year, 6 per quarter during the second year, 7 per quarter during the third year, and so on, ending at 12 per quarter in the eighth year. Assuming a nominal annual rate of 6% convertible quarterly, find the present value of this annuity.

Increasing Perpetuities

Three types of increasing perpetuities are introduced in the box below. The derivations for the present values of these annuities has been omitted.

Increasing Perpetuities

- **General Increasing Perpetuities:** Consider a perpetuity immediate that pays P at the end of the first year with subsequent payments increasing by Q per year. The present value of this perpetuity is given by $PV = \frac{P}{i} + \frac{Q}{i^2}$.
- **Standard Increasing Perpetuities:** Assume a perpetuity immediate pays 1 at the end of the first year with subsequent payments increasing by 1 per year. The present value of such a perpetuity is denoted by $(Ia)_{\overline{\infty|}}$ and is given by $(Ia)_{\overline{\infty|}} = \frac{1}{i\,d} = \frac{1}{i} + \frac{1}{i^2}$.
- **Increasing to Level Perpetuities**: Consider a perpetuity immediate paying 1, 2, 3, …, n for the first n years and paying n at the end of each subsequent year. The present value of this perpetuity is $PV = \frac{\ddot{a}_{\overline{n|}i}}{i}$.

Example 2.42 Find the present value of a perpetuity immediate that pays 9 at the end of the first year, with each subsequent payment increasing by 4. Assume an annual effective interest rate of 5%.

Example 2.43 Find the present value of a perpetuity-immediate with payments starting at 5, increasing by 5 each year until reaching 100, and then staying at 100 from then on. Assume that $i = 8\%$.

m-thly Increasing Annuities

We will consider two types of *m*-thly increasing annuities. The first type makes *m*-thly payments, but the payments increase only at the end of each year. The second type makes *m*-thly payments, with the increases also occurring on an *m*-thly basis. You should be familiar with the notation presented here, if not the actual formulas.

- The symbol $(Ia)^{(m)}_{\overline{n|}}$ represents the PV of an annuity immediate that makes level *m*-thly payments of k / m during year *k*. In other words, the annuity makes level *m*-thly payments of $1 / m$ during year 1, level *m*-thly payments of $2 / m$ during year 2, and so on. It can be shown that $(Ia)^{(m)}_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i^{(m)}}$.
- The symbol $(I^{(m)}a)^{(m)}_{\overline{n|}}$ represents the PV of an annuity immediate that makes *m*-thly payments, with the first payment equal to $1/m^2$, and with *each* subsequent payment increasing by $1/m^2$. It can be shown that $(I^{(m)}a)^{(m)}_{\overline{n|}} = \frac{\ddot{a}^{(m)}_{\overline{n|}} - nv^n}{i^{(m)}}$.

2.7 GEOMETRIC ANNUITIES

Geometric Annuities

In this section we will consider annuities in which the payments form a geometric, rather than arithmetic, sequence. In other words, the ratio between any two consecutive payments will be a constant.

Consider an n-year annuity-immediate in which the first payment is P, and each subsequent payment increases by a factor of $(1+k)$. That is to say, each payment is $k \times 100\,\%$ larger than the previous. The time diagram for such an annuity is shown below.

0	1	2	3	$\cdots$	n
	P	$P(1+k)$	$P(1+k)^2$	$\cdots$	$P(1+k)^{n-1}$

The present value of this annuity would be given by the following expression:

$$PV \;=\; Pv + P(1+k)v^2 + P(1+k)^2 v^3 + P(1+k)^3 v^4 + \ldots + P(1+k)^{n-1} v^n$$

The right-hand side of this equation can be rewritten as follows:

$$PV \;=\; \frac{P}{1+k}\left[(1+k)v + P(1+k)^2 v^2 + P(1+k)^3 v^3 + P(1+k)^4 v^4 + \ldots + P(1+k)^n v^n\right]$$

Let $v' = (1+k)v$. Substituting this value into the equation above gives us:

$$PV \;=\; \frac{P}{1+k}\left[v' + (v')^2 + (v')^3 + \ldots + (v')^n\right]$$

We could write the expression $\left[v' + (v')^2 + (v')^3 + \ldots + (v')^n\right]$ as $a_{\overline{n}|i'}$ if we could find an appropriate interest rate i'. In particular, we would need to find a rate i' such that $v' = \frac{1}{1+i'}$. Since $v' = (1+k)v = \frac{1+k}{1+i}$, this gives us $\frac{1}{1+i'} = \frac{1+k}{1+i}$. Solving this equation for i' gives us $i' = \frac{i-k}{1+k}$.

Thus, we conclude that the present value of the geometric annuity is given by $PV = \frac{P}{1+k} a_{\overline{n}|i'}$ where $i' = \frac{i-k}{1+k}$.

The recommended method of finding the accumulated value of this annuity is to first find the present value, and then accumulate this forward to time n. Note that you will need to use the true effective interest rate i, and not i', to accumulate the annuity forward.

Geometric Annuities

Consider an n-year annuity-immediate in which the first payment is P and each subsequent payment increases by a factor of $(1+k)$.

- The present value of this annuity at $t = 0$ is given by $PV = \frac{P}{1+k} \cdot a_{\overline{n}|i'}$ where $i' = \frac{i-k}{1+k}$.
- The accumulated value of this annuity at $t = n$ is given by $AV = \frac{P}{1+k} \cdot a_{\overline{n}|i'}(1+i)^n$

Example 2.44 Rusty purchases a 20-year annuity immediate. The first annuity payment is 1000 and the payments increase by 2% each year. Assuming an annual effective interest rate of 5%, find the price Rusty paid for this annuity.

It is also possible for payments in an annuity to decrease at a geometric rate. This results in a negative value for k.

Example 2.45 The first payment in an annuity immediate is equal to 2500. Each subsequent payment is 3% less than the previous payment. The payments continue for as long as their value is greater than 1500. At an annual effective interest rate of 6%, find the present value of this annuity.

As with arithmetically increasing annuities, we will use fusion to work with m-thly annuities for which the payments remain constant throughout the year, but increase in a geometric fashion at the end of each year.

Example 2.46 A loan of 275,000 will be repaid by monthly payments over the course of 20 years. The first payment is in the amount of P and occurs one month after the loan was made. Payments are level during any given year, but increase by 2.5% at the end of each year. Assuming an annual effective interest rate of 6%, find P.

Geometric Perpetuities

The special interest rate i' can be used to work with geometrically perpetuities as well as geometric annuities.

Geometric Perpetuities

Assume that a the first payment in an perpetuity immediate is equal to P, and each subsequent payment increases by a factor of $(1+k)$.

- The present value of this perpetuity at $t=0$ is given by $PV = \frac{P}{1+k} \cdot a_{\overline{\infty}|i'}$ where $i' = \frac{i-k}{1+k}$.

Example 2.47 A perpetuity immediate makes level payments of 100 for 8 years. After the first 8 years, each payment is 4% greater than the previous payment. Assuming an annual effective interest rate of 9%, calculate the present value of this perpetuity.

So far, we have only encountered examples in which the rate of increase in payments is strictly less than the annual effective interest rate. Notice that if $k = i$, then $i' = 0$, which results in $a_{\overline{n}|i'} = n$. If $k > i$, then $i' < 0$. The formulas provided will still work in this case. Just make sure to pay careful attention to the sign on i' when entering the rate into an annuity formula or into the BA II Plus calculator.

Example 2.48 An annuity immediate makes annual payments for 10 years. The first payment is equal to 50, and each subsequent payment is K% higher than the previous payment. Assuming an annual effective interest rate of 4%, calculate the present value of the annuity in each of the following cases: (a) $K = 2$, (b) $K = 4$, (c) $K = 6$

2.8 CONTINUOUS ANNUITIES

In this section, we will consider annuities which make payments continuously. Such annuities do not literally exist, but can serve as useful approximations for funds that make very frequent payments. To calculate the present value of such an annuity, we will need to know the rate at which the payments are made, as well as the continuous force of interest.

General Formulas for Continuous Paying Annuities

Assume that an annuity makes continuous payments to the owner of the annuity. The payments continue for n years and the rate of payment at time t is given by $f(t)$. Assume that interest is accumulated at a continuously changing force of interest δ_t which is associated with an accumulation function $a(t)$.

Consider an infinitesimal interval of time centered at time t and with length dt. The amount of money payed during this interval of time is $f(t)dt$. The present value of the payment $f(t)dt$ is equal to $f(t)dt / a(t)$. Summing the present values of all such payments over all possible times t results in the following integral for the total present value of the annuity: $PV = \int_0^n f(t) / a(t) dt$. We can derive a similar integral for the accumulated value of such an annuity, but it would be more complicated. It is generally simpler to calculate the present value and then accumulate that forward to find the accumulated value.

General Formulas for Continuous Paying Annuities

Assume an n-year annuity makes continuous payments at a rate of $f(t)$ at time t. Assume that interest is accumulated at a force of interest δ_t which results in an accumulation function $a(t)$.

- The present value of this annuity at $t = 0$ is given by $PV = \int_0^n \frac{f(t)}{a(t)} dt$.
- The accumulated value of this annuity at $t = n$ is given by $AV = a(n) \cdot PV$.

Example 2.49 An 8-year annuity makes continuous payments at a rate of $5t$ at time t. Assume a continuous force of interest given by $\delta_t = 0.2t / (1 + 0.1t^2)$.

a) Find the present value of this annuity.
b) Find the accumulated value of this annuity at time $t = 8$.

Example 2.50 Payments are made to an account at a continuous rate of $(6k + 6k)$. Assume a continuous force of interest given by $\delta_t = 1 / (8 + t)$. And the end of the fifth year, the account is worth 12,000. Find k.

Example 2.51 An annuity makes continuous payments at a rate $6t^2$ for 10 years. The price of this annuity is determined using a constant force of interest $\delta = 0.10$. Find the price of the annuity.

We will now consider several special cases of continuous-paying annuities.

Constant Payments and Constant Force of Interest

Consider an annuity that makes continuous payments at a constant rate of $f(t) = 1$ per year. Assume that the force of interest is given by a constant δ. The present value of such an annuity is denoted by the symbol $\bar{a}_{\overline{n}|}$ and the accumulated value is denoted by $\bar{s}_{\overline{n}|}$.

Since δ is constant, we are working with compound interest and $a(t) = e^{\delta t} = (1+i)^t$, where i is the associated annual effective rate of interest. We also note that since we are working with compound interest, the present value factor $1/a(t)$ can be written as $1/a(t) = v^t = e^{-\delta t} = (1+i)^{-t}$.

Substituting $f(t) = 1$ and $1/a(t) = v^t$ into the general present value formula for a continuous-paying annuity yields the integral $\bar{a}_{\overline{n}|} = \int_0^n v^t\,dt$. Solving this integral provides us with the formula $\bar{a}_{\overline{n}|} = \dfrac{1 - v^n}{\delta}$.

By accumulating this expression forward, we see that the accumulated value of this annuity is $\bar{s}_{\overline{n}|} = \dfrac{(1+i)^n - 1}{\delta}$.

Continuous Paying Annuity with Constant Payments and Constant Force of Interest

Assume an n-year annuity makes continuous payments at a rate of 1 per year. Assume that interest is accumulated at a constant force of interest δ.

- The present value of this annuity at $t = 0$ is given by $\bar{a}_{\overline{n}|} = \dfrac{1 - v^n}{\delta}$.
- The accumulated value of this annuity at $t = n$ is given by $\bar{s}_{\overline{n}|} = \dfrac{(1+i)^n - 1}{\delta}$.

Notice that the formulas for $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$ are very similar to those for $a_{\overline{n}|}$ and $s_{\overline{n}|}$. The only difference is that the i in the denominator of the standard annuity formulas is replaced with δ for continuous-paying annuities.

If an annuity pays R each year, paid continuously over the course of the year, then its present and accumulated values are given by $R\bar{a}_{\overline{n}|}$ and $R\bar{s}_{\overline{n}|}$, respectively.

Example 2.52 A 20-year annuity makes continuous payments at a rate of 8 per year. Assume $i = 5\%$.

a) Find the present value of this annuity at $t = 0$.
b) Find the accumulated value of this annuity at $t = 20$.

Example 2.53 A 12-year annuity makes continuous payments at a rate of 3 per year. In addition to the continuous payments, discrete payments of 2 are made at the end of each year. The effective annual rate of interest is 6%.

a) Find the present value of this annuity at $t = 0$.
b) Find the accumulated value of this annuity at $t = 12$.

Example 2.54 You are given that $\bar{a}_{\overline{12}|} = 6.988$ and $\dfrac{d}{d\delta}\left(\bar{a}_{\overline{10}|}\right) = -33.737$. Find δ.

Notice that the equation $\bar{a}_{\overline{12}|} = 6.988$ in the previous example has only one unknown in it: δ. This problem can be solved quickly by using the Table function in the TI-30X to plug several different values of δ into $\bar{a}_{\overline{12}|}$.

Increasing Continuous Annuities

We will now consider the special case of a continuous-paying annuity in which the rate of payment increases linearly over time and the force of interest is constant. To that end, assume that $f(t) = t$ and let δ denote the constant force of interest. The present value of such an annuity is denoted by $(\bar{I}\,\bar{a})_{\overline{n}|}$ and its accumulated value is denoted by $(\bar{I}\,\bar{s})_{\overline{n}|}$.

Substituting $f(t) = t$ and $1/a(t) = v^t$ into the general present value formula for a continuous-paying annuity yields the integral $(\bar{I}\,\bar{a})_{\overline{n}|} = \int_0^n t\,v^t\,dt$. Solving this integral gives us $(\bar{I}\,\bar{a})_{\overline{n}|} = \dfrac{\bar{a}_{\overline{n}|} - n\,v^n}{\delta}$.

By accumulating this expression forward, we see that the accumulated value of this annuity is $(\bar{I}\,\bar{s})_{\overline{n}|} = \dfrac{\bar{s}_{\overline{n}|} - n}{\delta}$.

Continuous Paying Annuity with Increasing Payments and Constant Force of Interest

Assume an n-year annuity makes continuous payments at a rate of $f(t) = t$ and interest is accumulated at a constant force of interest δ.

- The present value of this annuity at $t = 0$ is given by $(\bar{I}\,\bar{a})_{\overline{n}|} = \dfrac{\bar{a}_{\overline{n}|} - n\,v^n}{\delta}$.
- The accumulated value of this annuity at $t = n$ is given by $(\bar{I}\,\bar{s})_{\overline{n}|} = \dfrac{\bar{s}_{\overline{n}|} - n}{\delta}$.

If an annuity makes continuous payments at a rate of $f(t) = k\,t$, then its present and accumulated values are given by $k(\bar{I}\,\bar{a})_{\overline{n}|}$ and $k(\bar{I}\,\bar{s})_{\overline{n}|}$, respectively.

Example 2.55

A 12-year annuity makes continuous payments at a rate of $4t$. Assume $i = 6\%$.

a) Find the present value of this annuity at $t = 0$.

b) Find the accumulated value of this annuity at $t = 12$.

Example 2.56

At an annual effective rate of 7%, the following annuities have the same present value:

i) A 10-year annuity that makes continuous payments at a constant rate of 8 per year.

ii) A 10-year annuity that makes continuous payments at a rate of $k\,t$ at time t.

Find k.

Decreasing Continuous Annuities

Next we will look at continuous-paying annuities in which the rate of payment decreases linearly over time and the force of interest is constant. Assume that $f(t) = n - t$ and let δ denote the constant force of interest. The present value of such an annuity is denoted by $(\bar{D}\bar{a})_{\overline{n|}}$ and its accumulated value is denoted by $(\bar{D}\bar{s})_{\overline{n|}}$.

Substituting $f(t) = n - t$ and $1 / a(t) = v^t$ into the general present value formula for a continuous-paying annuity yields the integral $(\bar{D}\bar{a})_{\overline{n|}} = \int_0^n (n - t)v^t dt$. Solving this integral results in the formula $(\bar{D}\bar{a})_{\overline{n|}} = \dfrac{n - \bar{a}_{\overline{n|}}}{\delta}$. Accumulating this expression forward, gives us that $(\bar{D}\bar{s})_{\overline{n|}} = \dfrac{n(1+i)^n - \bar{s}_{\overline{n|}}}{\delta}$.

Continuous Paying Annuity with Decreasing Payments and Constant Force of Interest

Assume an n-year annuity makes continuous payments at a rate of $f(t) = n - t$ and interest is accumulated at a constant force of interest δ .

- The present value of this annuity at $t = 0$ is given by $(\bar{D}\bar{a})_{\overline{n|}} = \dfrac{n - \bar{a}_{\overline{n|}}}{\delta}$.
- The accumulated value of this annuity at $t = n$ is given by $(\bar{D}\bar{s})_{\overline{n|}} = \dfrac{n(1+i)^n - \bar{s}_{\overline{n|}}}{\delta}$.

If an annuity makes continuous payments at a rate of $f(t) = k(n - t)$, then its present and accumulated values are given by $k(\bar{D}\bar{a})_{\overline{n|}}$ and $k(\bar{D}\bar{s})_{\overline{n|}}$, respectively.

Example 2.57 A 15-year annuity makes continuous payments. The rate of payment is equal to 60 at $t = 0$, and decreases linearly until it reaches 0 at $t = 15$. The annual effective interest rate is 8%.

a) Find the present value of this annuity at $t = 0$.
b) Find the accumulated value of this annuity at $t = 15$.

Example 2.58 An annuity makes continuous payments for 30 years. Payments are made at a constant rate of 50 per year for the first 20 years. During the last 10 years, the rate of payment decreases linearly from 50 to 0. Find the present value of this annuity at an annual effective rate of interest is 7%.

2.9 MISCELLANEOUS ANNUITY TOPICS

We will conclude this chapter with a discussion of some miscellaneous topics related to annuities.

The $a_{\overline{kn}|} / a_{\overline{n}|}$ Formula

Notice that $a_{\overline{2n}|} = v + v^2 + ... + v^n + v^{n+1} + v^{n+2} + ... + v^{2n} = [v + v^2 + ... + v^n] + v^n[v + v^2 + ... + v^n] = a_{\overline{n}|}(1+v^n)$. It follows that $a_{\overline{2n}|} / a_{\overline{n}|} = 1 + v^n$. Similar formulas can be derived for $a_{\overline{kn}|} / a_{\overline{n}|}$ where $k > 2$.

The $a_{\overline{kn}|} / a_{\overline{n}|}$ Formulas

The following identities hold for all values of n and i.

- $\dfrac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$
- $\dfrac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n}$
- $\dfrac{a_{\overline{4n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n} + v^{3n}$

These formulas can be used to simplify the algebra involved in certain types of problems.

Example 2.59 At an annual effective interest rate i, the following annuities have the same present value.

i. A 12-year annuity immediate with level annual payments of 27.
ii. A 24-year annuity immediate with level annual payments of 20.

Find i.

Example 2.60 Assuming an annual effective interest rate i, you are given:

i. The present value of an n-year annuity due with annual payments of 100 is 1030.61.
ii. The present value of a $2n$-year annuity due with annual payments of 150 is 2099.24.

Find i.

Example 2.61 You are given $\ddot{a}_{\overline{n}|i} = 15.2191$ and $\ddot{a}_{\overline{2n}|i} = 21.1663$. Find n.

Example 2.62 An annuity immediate pays X per year for $3n$ years. The present value of the annuity is 1000. The present value of the first n payments is equal to 640. Find the present value of the last n payments.

Example 2.63 Assuming an annual effective interest rate i, you are given the following information:

i. The present value of an $2n$-year annuity immediate that pays 4 for the first n years and 3 for the last n years is 34.8113.
ii. The present value of an n-year deferred annuity immediate that pays 3 for n years is equal to 6.9728

Find i.

Selling Annuities and Early Repayment

Assume an individual purchases an n-year annuity immediate that pays R per year at an annual effective interest rate of i. Let P denote the price paid by the individual. In other words, P is the present value of the annuity at the annual effective rate of i.

Now assume that after receiving the first m payments, the individual sells the rights to receive the remaining $n-m$ payments to another party for a price of Q. Depending on the value of Q, the original owner of the annuity and the new purchaser could each realize interest rates that are different from each other, and both different from i. Let j denote the rate realized by the original owner of the annuity and let k denote the rate earned by the purchaser of the final $n-m$ payments. These rates can be determined as follows:

- The rate for the original owner is determined by the equation $P = R\,a_{\overline{m}|j} + Q v^{-m}$.
- The rate for the new purchaser is determined by the equation $Q = R\,a_{\overline{n-m}|k}$.

For most values of m, it is practically impossible to solve for the rate in an expression such as $P = R\,a_{\overline{m}|j} + Q v^{-m}$ without using some form of technology. Fortunately, the TVM feature of the BA II Plus calculator can be used to solve such problems.

As an example, assume that we wish to solve for j in the equation $100 = 12\,a_{\overline{8}|j} + 90\,v^{8}$. We could do so by entering the following information into the BA II Plus:

- [2ND] [CLR TVM] 8 [N] 100 [PV] 12 [+/-] [PMT] 90 [+/-] [FV] [CPT] [I/Y]

Example 2.64

Arthur borrows 2000 from Betty. Arthur agrees to repay the loan over the course of 10 years by making annual payments at an annual effective interest rate of 7%.

Immediately after receiving the sixth payment from Arthur, Betty sells the rights to receive the remaining four payments to Chad for a price of 1000. The size of the payments do not change, but Arthur now pays them to Chad.

a) Determine annual effective rate of interest earned by Chad.
b) Determine annual effective rate of interest earned by Betty.
c) Determine annual effective rate of interest paid by Arthur.

The approach above can also be used to solve for the interest rate in problems involving early repayment of a loan.

Example 2.65

Jorge borrows 160,000 at a nominal annual rate of interest of 4.8% convertible monthly. He agrees to repay the loan by making monthly payments of R.

At the end of the 15th year, immediately after making payment number 180, Jorge repays the remaining balance of the loan. He is also charged an early repayment penalty of 8000 at this time. Find the rate of interest actually paid by Jorge. Express your answer as a nominal annual rate of interest compounded monthly.

Varying Rates for Annuities

It is possible to encounter an annuity problem in which the effective rate of interest changes at some point during the duration of the annuity. In calculating the present or accumulated value of such an annuity, you will generally need to split the annuity into separate annuities such that the rate of interest is consistent within the time period spanned by each annuity. Be careful about using the correct rate of interest when discounting or accumulating through any particular time period.

For example, assume that we wish to find the present value of a 15-year annuity immediate with payments of 1 assuming an annual effective rate of 4% during years 1 – 10 and an annual effective rate of 6% during years 11 – 15. We can calculate the present value as follows:

1. Discount payments 11 – 15 to time 10 using $a_{\overline{5}|6\%}$.
2. Discount $a_{\overline{5}|6\%}$ through years 1 – 10 to time 0 by multiplying by $(1.04)^{-10}$.
3. Discount payments 1 – 10 to time 0 using $a_{\overline{10}|4\%}$. Add this to the result from Step 2.

The resulting present value is: $PV = a_{\overline{10}|4\%} + (1.04)^{-10} a_{\overline{5}|6\%}$.

Example 2.66 Anita deposits 60 into a fund at the end of each year for 20 years. The fund earns an annual effective interest rate of 5% for the first 14 years and an annual effective interest rate of 8% during the last 6 years.

a) Find the accumulated value of the fund at the end of the 20 years.
b) Find the overall annual effective rate of interest realized by Anita during the 20 year period.

Continuous Force of Interest, but Discrete Payments

Assume that you are asked to find the present value or accumulated value of an annuity that makes discrete payments, but you are given a continuous force of interest, δ_t . In this scenario, you can use the following formulas for $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

- $a_{\overline{n}|} = \frac{1}{a(1)} + \frac{1}{a(2)} + \frac{1}{a(3)} + \ldots + \frac{1}{a(n)}$
- $s_{\overline{n}|} = a(n)\left[\frac{1}{a(1)} + \frac{1}{a(2)} + \frac{1}{a(3)} + \ldots + \frac{1}{a(n)}\right]$

Example 2.67 Assume interest is credited according the the force of interest $\delta_t = \frac{2}{5+t}$, find $a_{\overline{4}|}$ and $s_{\overline{4}|}$.

Example 2.68 Erin makes deposits of K into an account at the end of each year for for 5 years. The account earns interest at a force of interest given by $\delta_t = \frac{0.2\,t}{1 + 0.01t^2}$. Determine the annual effective rate of interest that Erin earned over the course of the 5 years.

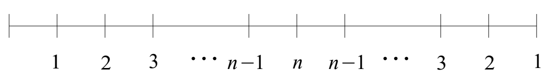
Palindromic Annuities

Consider an annuity immediate that makes annual payments that start at 1, increasing by 1 each year until they reach n, and then decreasing by 1 each year until they again reach 1. The payments in this annuity follow a palindromic pattern. The present value of this annuity can be calculated by treating it as an n-year increasing annuity plus an $(n-1)$-year decreasing annuity that is deferred by n years. Such an approach would yield a present value of $PV = (Ia)_{\overline{n}|} + v^n (Da)_{\overline{n-1}|}$. We will consider another approach to calculating the present value of such an annuity.

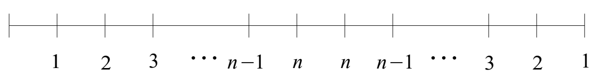
We first split the annuity into n level annuities, each of which pays 1 for n years. The time of the first payment for these n annuities will vary from 1 to n. You should convince yourself that this produces the same total payment at the end of each year as the original annuity.

We now calculate the present value of each sub-annuity at the time of its own first payment. This produces a set of n payments of $\ddot{a}_{\overline{n}|}$ at times 1, 2, …, n. Calculating the total present value of these payments results in the formula $PV = a_{\overline{n}|} \cdot \ddot{a}_{\overline{n}|}$.

We can take a similar approach to calculate the present value of a palindromic annuity that makes two consecutive payments of n before beginning to decrease. Time diagrams both types of palindromic annuities are shown below, along with formulas for their present values.

- 

  $$PV = a_{\overline{n}|} \cdot \ddot{a}_{\overline{n}|} = (1+i)\left[a_{\overline{n}|}\right]^2$$

- 

  $$PV = a_{\overline{n}|} \cdot \ddot{a}_{\overline{n+1}|} = (1+i)\left[a_{\overline{n}|} a_{\overline{n+1}|}\right]$$

Example 2.69

Assuming an annual effective interest rate of 5%, calculate the present value of each of the annuities described below.

a) An 11-year annuity immediate with payments of 1, 2, 3, 4, 5, 6, 6, 4, 3, 2, 1.
b) A 12-year annuity immediate with payments of 1, 2, 3, 4, 5, 6, 6, 5, 4, 3, 2, 1.
c) An 11-year annuity due with payments of 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1.
d) A 12-year annuity due with payments of 1, 2, 3, 4, 5, 6, 6, 5, 4, 3, 2, 1.

Example 2.70

At an annual effective interest rate of i, the following annuities have the same present value:

i) A 9-year annuity-due with payments of 1, 2, 3, 4, 5, 4, 3, 2, 1.
ii) An 8-year annuity-immediate with payments of k, $2k$, $3k$, $4k$, $4k$, $3k$, $2k$, k, where $k = 1.4$