CHAPTER 3 – Reinvestment and Amortization

3.1 REINVESTMENT

Effective Yield of an Annuity Without Reinvestment

Assume a loan of L accumulates interest at an annual effective interest rate of i and is to be repaid over the course of n years. Let's consider two possible methods by which the borrower could repay the loan:

1. The borrower could make a single lump sum payment of $L(1+i)^n$ at the end of n years.
2. The borrower makes annual payments of R at the end of each year. The size of these payments can be determined using the formula $L = Ra_{\overline{n}|i}$.

The total amount that the lender receives during the n-year period is smaller in the second repayment plan than in the first. However, under the second payment plan, the lender will receive payments throughout the lifetime of the loan and can thus use this money to pursue other investments. If, however, the lender does not opt to reinvest the payments as they are received, then those payments will fail to earn interest after they are received. As a result, the lender's overall yield rate during the n-year period will be smaller than i. Let's illustrate this concept with an example.

- Assume you pay ABC Investments 421.24 to purchase a 5-year annuity-immediate with payments of 100.
- Since $421.24 = 100a_{\overline{5}|6\%}$, your yield on this investment is 6%.
- Notice that the sum of the payments you received is 500. However, the accumulated value of 421.24 invested for 5 years at 6% is $421.24(1.06)^5 = 563.71$, which is clearly greater than 500.
- As ABC makes payments to you, it no longer has to pay interest on the balance of those payments, which explains the apparent discrepancy in the previous bullet point. If no payments are made to you during the 5 years, then you would earn 6% interest on the entire balance of 421.25 for the 5 year period, thus yielding 563.71.
- Assume that we adopted an investment strategy of purchasing this annuity from ABC Investments and depositing the annuity payments under a mattress as they arrive. Our initial investment would be 421.24, and we would "cash out" for 500 at the end of 5 years. To find the effective yield on our investment strategy over the 5 years, we solve $421.24(1+i)^5 = 500$, which gives $i = 3.4875\%$.
- Even though it is true that the annuity we purchased from ABC yielded 6%, our overall investment strategy ultimately earned only 3.4875% since the payments received were not reinvested.

Reinvesting Annuity Payments

Assume that L is paid for an annuity that makes level payments of R at the end of each year for n years. Let i be the yield rate on the annuity. Then $L = Ra_{\overline{n}|i}$. Suppose that as annuity payments are received, they are reinvested into Account X, which earns interest at an annual effective rate of j. Since R is deposited into Account X at the end of each year, the value of this account at the end of n years will be $Rs_{\overline{n}|j}$.

This investment strategy required an initial investment of L, and yields an amount of $Rs_{\overline{n}|j}$ at the end of n years. The effective rate of return on this strategy, k, can thus be found by solving $L(1+k)^n = Rs_{\overline{n}|j}$.

Example 3.1

Kyle pays 3000 for a 10-year annuity immediate with annual payments of 400. As the payments are received, they are reinvested into an account that earns an annual effective rate j. Kyle's overall effective rate of interest earned is 4.8176%. Find j.

Reinvesting Interest Payments

- Assume that an amount of L is loaned for a period of n years at an annual effective rate i.
- Suppose that interest payments of Li are made at the end of each year, and the original amount of L is paid back at the end of year n.
- If the interest payments are reinvested into an account earning a rate of j, then the balance of this account at the end of n years will be $Lis_{\overline{n}|j}$.
- At time n, the lender will receive the principal repayment of L, plus the AV of the reinvestment account, which is $Lis_{\overline{n}|j}$.
- The lender's overall yield k can be found by solving $L(1+k)^n = Lis_{\overline{n}|j} + L$.

Example 3.2

Kara loans David 2000 at an annual effective rate of i. David makes annual interest payments at the end of each year, and repays the 2000 at the end of year 8. Kara reinvests the interest payments received from David into an account that earns 3% annual effective. Kara's overall rate of return over the course of the 8 years was 6.236%. Find i.

Example 3.3

Peter loans Harry 2400 at 8% annual effective. At the end of each year, Harry repays the interest accumulated over the course of the previous year, plus an additional 200. The loan is repaid after 12 such payments. As Peter receives payments from Harry, he reinvests them into Fund X, which earns interest at an annual effective rate of 5%. Find the rate of return earned by Peter over the course of the 12 years.

Multiple Reinvestment Accounts

You will occasionally encounter problems with multiple reinvestment funds. Dealing with such problems requires a combination of the methods discussed above.

Example 3.4

An investor purchases a bond for 1000. The bond will make "coupon payments" of 100 at the end of each year for 12 years, and a principal repayment of 1000 at the end of the twelfth year.

The coupon payments are reinvested into Fund X, which earns interest at an annual effective rate of 5%. At the end of each year, the accumulated interest from Fund X is deposited into Fund Y, which earns 3% annual effective.

Determine the investors rate of return over the 12 year period.

Example 3.5

Jesse pays P for an annuity that makes payments of 160 at the beginning of each year for 20 years. The annuity payments are reinvested into Fund X, which earns 10% annual effective. The interest earned by Fund X each year is withdrawn and deposited into Fund Y, which earns 6% annual effective. Jesse's annual yield rate over the 20 year period is 8%. Find P.

3.2 AMORTIZING A LOAN

Amortization is the process of settling a debt. Depending on the terms set when a loan is made, there are many different methods that can be used to repay the debt. The following example compares three possible methods.

Example 3.6

Pam, Cheryl, and Ray each borrow 2000 for 10 years at an annual effective rate of 6%. Pam makes no payments until the end of the 10 years, at which point she repays the loan with one lump sum payment. Cheryl pays the accumulated interest at the end of each year and repays the principal at the end of the 10 years. Ray repays the loan by making 10 level annual payments at the end of each year. Calculate the amount of interest paid by each of the three individuals.

Amortization Using A Level Annuity

Throughout the rest of this section, we will consider only cases in which debt is amortized using level annuities. Before looking at examples, we need to establish some notation and terminology. For simplicity, we will assume here that the payments occur on an annual basis. In general, the payment periods in am amortization problem could be quarters, months, weeks, or any other period of time.

- Let L represent the original **loan amount**. This is also called the **initial principal**.
- Let R be the level annual **payment**. Then $L = R\,a_{\overline{n|}}$ and $R = L \,/\, a_{\overline{n|}}$.
- Let B_t be the amount owed at time t. This quantity is referred to as the **unpaid balance** or the **outstanding principal** at time t. Note that $B_0 = L$.

Calculating Unpaid Balance

We will make frequent use of two different algebraic methods for calculating the unpaid balance of a loan at time t. These methods are called the *retrospective method* and the *prospective method*.

- **Retrospective Method.** Assume that no payments have been made against the debt. Then the amount owed at time t would be $L(1+i)^t$. If annual payments of R are made, however, then the outstanding balance would be reduced by the accumulated value of these payments. That is: $B_t = L(1+i)^t - R\,s_{\overline{t|}}$
- **Prospective Method.** Regardless of the number of payments that have been made up until this point, and regardless of the original amount of the loan, the currently outstanding balance must be the present value of all future payments that have yet to be made. Thus: $B_t = R\,a_{\overline{n-t|}}$.

Example 3.7

Doug borrows 50,000 at 6% convertible monthly. According to the original terms of the loan, the debt is to be repaid with level payments at the end of each month for 20 years, with no option for early repayment. At the end of 8 years, Doug renegotiates the terms of the loan. Under the new terms, he will pay the remaining balance with monthly payments lasting 6 more years, but his debt will now accumulate interest at 6.3% convertible monthly. Calculate the total amount of money that Doug saved by renegotiating the debt.

Calculator Tip: In the previous problem, the unpaid balance B_{96} can be calculated using the BA II as follows:

- [2ND] [CLR TVM] 240 [N] 0.5 [I/Y] 50000 [PV] [CPT] [PMT] 96 [N] [CPT] [FV]

Example 3.8

Cedric takes out a loan that is charged interest at an annual effective rate of 4%. He agrees to pay the loan back over the course of 30 years by making level payments at the end of each year. After 12 years, Cedric refinances his loan to obtain a lower rate. Under the terms of the refinance, he makes an immediate payment of 20,000 which is applied to the loan balance. His rate on the remaining balance is then lowered to 3%. Under the new terms of the loan, Cedric's annual payments for the remaining 18 years are 8127.08. Find the original loan amount.

Calculator Tip: The previous problem can be calculated using the BA II as follows:

- [2ND] [CLR TVM] 18 [N] 3 [I/Y] 8127.08 [+/-] [PMT] [CPT] [PV] [+] 20000 [=] [PV] 4 [I/Y] [CPT] [PMT] 30 [N] [CPT] [PV]

Amortization Tables

When amortizing a loan, each payment can be split into two pieces: the interest payment I_t and the principal reduction P_t. The interest portion is equal to the interest that has been accumulated since the last payment (i.e. $I_t = i \cdot B_{t-1}$). The principal portion of the payment is the amount by which the unpaid balance is reduced after the interest is paid (i.e. $P_t = R - I_t$).

An amortization table is a table that displays the values R, I_t, P_t, and B_t for each payment. Consider a loan with the following parameters: $L = 1000$, $R = 250$, $n = 5$, $i = 7.9308\%$. The amortization table for this loan is provided on the left below. The table to the right is the amortization table for an arbitrary loan with $n = 5$. We will use this general table to help us obtain formulas for directly calculating I_t and P_t without having to construct an amortization table.

t	R_t	I_t	P_t	B_t
0				1000
1	250	79.31	170.69	829.31
2	250	65.77	184.23	645.08
3	250	51.16	198.84	446.24
4	250	35.39	214.61	231.63
5	250	18.37	231.63	0
Totals	**1250**	**250**	**1000**	

t	R_t	I_t	P_t	B_t
0				$R\,a_{\overline{5}\rceil}$
1	R	$R(1-v^5)$	$R\,v^5$	$R\,a_{\overline{4}\rceil}$
2	R	$R(1-v^4)$	$R\,v^4$	$R\,a_{\overline{3}\rceil}$
3	R	$R(1-v^3)$	$R\,v^3$	$R\,a_{\overline{2}\rceil}$
4	R	$R(1-v^2)$	$R\,v^2$	$R\,a_{\overline{1}\rceil}$
5	R	$R(1-v^1)$	$R\,v^1$	0
Totals	$5R$	$5R - L$	L	

Interest and Principal Payments

We can use the table on the right above to make the following general observations about I_t and P_t:

- $I_t = R(1-v^{n+1-t})$ and $P_t = R\,v^{n+1-t}$.
- The values of P_t form a geometric sequence with common ratio $(1+i)$.
- $L = \Sigma P_t$.

Example 3.9

A loan of 5000 collects interest at an annual effective rate of 5% and is to be repaid with annual payments made over 12 years. Find the amount of interest paid and the principal repaid in the fifth installment.

Summary of Formulas

We summarize the formulas used for amortization in the table below.

Quantity	Definition	Formula 1	Formula 2
L	Original loan amount.	–	–
R	Level payment.	$R = L / a_{\overline{n}\vert}$	–
B_t	Unpaid balance at time *t*.	$B_t = L(1+i)^t - R s_{\overline{t}\vert}$	$B_t = R\, a_{\overline{n-t}\vert}$
I_t	Interest portion of payment *t*.	$I_t = i \cdot B_{t-1}$	$I_t = R\left(1 - v^{n+1-t}\right)$
P_t	Principal portion of payment *t*.	$P_t = R - I_t$	$P_t = R v^{n+1-t}$

It is also important to note that the values of P_t form a geometric sequence with ratio $(1+i)$, and that $L = \Sigma P_t$.

Example 3.10 Gabe has a loan that is to be repaid with annual payments of 1000 at the end of each year for $2n$ years. The loan collects interest at an annual effective rate of 5.9%. The sum of the interest paid in year 1 plus the interest paid in year $n+1$ is equal to 1610. Find the amount of interest paid in year 8.

Example 3.11 Cora is repaying a loan by making payments of 2000 at the end of each quarter. The loan collects interest at a nominal rate of 8% convertible quarterly. The amount of interest paid in the tenth payment is 1271.51. Find the principle repaid with payment number 24.

Example 3.12 Bruce repays a loan by making payments at the end of each year for n years. The unpaid balance of the loan accumulates interest at a rate of 8% annual effective. The amount of interest paid in the final installment is 62.48. The total principal repaid at the time of the second-to-last payment is 6438.29. Calculate the principal repaid in the first payment.

Example 3.13 A loan is repaid over 15 years with level annual payments. The loan collects interest at 7% annual effective. The principal repaid with the fifth payment is 187. Find the loan amount.

Example 3.14 ABC Corp. borrowed 100,000 at a nominal rate of 6% convertible semiannually. The loan is to be repaid with level payments at the end of each six month period. The amount of interest paid in the eighth payment is 2516.82. Find the principle repaid with the fifteenth payment.

3.3 SINKING FUNDS

Assume that a borrower agrees to make interest payments on a loan at the end of each period. Since these interest payments do not repay any of the principal, the unpaid balance is once again equal to the original loan amount after each payment. Assume also that the borrower agrees to repay the original loan amount at some specified future time.

Suppose now that in addition to making the interest payments to the lender, the borrower also makes annual deposits into a side fund with the intent of eventually using the accumulated value of the side fund to repay the loan. A fund such as this side fund is called a **sinking fund**. Two reasons why the borrower might opt to repay a loan using the sinking fund method as opposed to the standard amortization method are: (1) the terms of the loan might not allow for payments (other than the last) to cover anything more than the interest, and (2) the sinking fund method is preferable to the amortization method if the sinking fund earns interest at a rate larger than what the original loan is being charged.

Before looking at examples, we will establish some notation and terminology relating to sinking funds.

- Let L be the original loan amount.
- Let i be the annual effective rate for the loan. Let j be the annual rate earned by the sinking fund.
- The size of the annual interest payments made to the lender are fixed at $I = i \cdot L$.
- The size of the annual sinking fund deposits are given by $SFD = L / s_{\overline{n}|j}$.
- The total amount paid by the borrower each year is $R = I + SFD$.
- The balance of the sinking fund at time t is given by $SFB_t = SFD\, s_{\overline{t}|j}$.

Example 3.15

A loan of 1000 is charged interest at 4% annual effective. The loan must be repaid in 10 years.

a) Assume that the loan is repaid using the standard amortization method by making level annual payments. Find the size of the payments.

b) Assume that the borrower makes interest payments to the lender at the end of each year and repays the original loan amount of 1000 at the end of year 10. The borrower also makes annual deposits into a sinking fund earning 6% annual effective in order to accumulate the 1000 to be repaid at time 10. Calculate the total amount paid each year by the borrower (including the interest payment and the sinking fund deposit).

c) Taking into account the effect of the sinking fund, determine the effective rate of interest that the borrower paid in the scenario outlined in Part (b).

Example 3.16

Julie borrows 20,000 for 18 years at an annual effective interest rate of i . She repays the loan using the sinking fund method. Her sinking fund earns an annual effective rate of 8%. Julie's total annual payment, including her interest payment and her sinking fund deposit, is equal to P . Had the effective rate on her loan been $2i$, then her total payment would have been $1.7P$. Find i .

Example 3.17 A loan of 60,000 is to be repaid over the course of 20 years. The borrower pays interest on the loan at the end of each year at a rate of 8%. The borrower also makes annual deposits into a sinking fund earning 6% with the intent of accumulating 60,000 in the sinking fund by the end of year 20. At the end of year 8, the rate earned by the sinking fund drops to 5%. Calculate the size of the sinking fund deposit for years 9 through 20.

Net Balance and Net Interest For the Sinking Fund Method

Under the sinking fund method, the size of the interest payments to the lender are level throughout the lifetime of the loan and the unpaid balance of the loan is the same after every payment. However, the balance of the sinking fund itself increases over time, as does the amount of interest earned by the sinking fund. This observation leads us to consider the concepts of net interest and net balance.

- The **net interest paid** at time t is denoted by I_t and is equal to the level interest payment minus the interest earned by the sinking fund. Thus, $I_t = I - j \cdot SFB_{t-1}$.
- The **net unpaid balance** at time t is denoted by B_t and is equal to the original loan amount minus the accumulated value of the sinking fund. Thus, $B_t = L - SFB_t$.

Example 3.18 A loan of 50,000 is repaid over 12 years by making annual interest payments at an effective rate of i, as well as level payments into a sinking fund earning 5% annual effective. The net interest paid during year 4 is 3400. Find the net interest paid during year 8.

Example 3.19 Trevor borrows 22,000 to be repaid in 10 years. He makes annual interest payments at 8% annual effective. Trevor also makes annual deposits of 1675 into a sinking fund earning an effective rate of i in order to accumulate 22,000 to repay the loan at the end of 10 years. Find Trevor's net balance at the end of year 7.

Example 3.20 Joanna will repay a loan over 12 years by making annual interest payments, as well as deposits of X into a sinking fund at the end of each year. The amount of interest earned by the sinking fund during the fourth year is $0.157625X$. The net amount of the loan immediately after the eighth payment is 2493.04. Find X .

3.4 VARYING PAYMENTS AND EQUAL PRINCIPAL REPAYMENT

Under the standard amortization method, as well as the sinking fund method, periodic payments are level. It is possible to establish a schedule for repaying a loan that utilizes varying payments instead. Consider that the payments could vary in any number of ways, it is not possible to establish general formulas to cover all such situations. When dealing with varying payments, one must apply general interest theory and annuity principles.

Example 3.21 A loan is to be repaid over five years with payments at the end of each month. The loan collects interest at a nominal rate of 6% convertible monthly. The first payment is 1000, and each later payment is 2% lower than the one preceding it. Find the unpaid balance at the end of year 3.

Example 3.22 Todd borrows X at an annual effective rate of i. The loan is to be repaid with payments at the end of each year for 14 years. The first payment is 700 and each subsequent payment decreases by 50. The amount of principal repaid in year 4 is equal to 325. Find X.

Example 3.23 A loan of 60,000 accumulates interest at an annual effective rate of 6%. The loan is to be repaid with payments at the end of each year for 32 years. The initial payment is X and each subsequent payment is X larger than the preceding payment. Find the amount of principal outstanding after payment number 19.

Equal Principal Repayment

One common method of utilizing varying payments to amortize a loan involves setting payments in such a way that an equal amount of principal is repaid with each installment. The details of this method are given below.

- Assume a loan of L accumulates interest at an annual effective rate of i.
- The loan is to be repaid with payments at the end of each year for n years.
- Denote the annual payment by R_t.
- We assume that each payment R_t repays an equal amount of principal given by $P_t = L / n$.
- Since the outstanding principal decreased by L / n with each payment, we have that $B_t = \frac{n-t}{n} L$.
- The amount of interest accumulated at the end of each year is thus given by $I_t = i \cdot B_{t-1} = \frac{n+1-t}{n} L \cdot i$.
- Combining some of the results from above, we see that $R_t = P_t + I_t = \frac{L}{n} + \frac{n+1-t}{n} L \cdot i$.
- Since L must be the PV of the loan payments, if follows that $L = \frac{L}{n} \cdot a_{\overline{n}|} + \frac{L\,i}{n} \cdot (Da)_{\overline{n}|}$.

Example 3.24 Ani borrowed 5000 from ABC Loans at 6% annual effective. The loan is to be repaid with payments at the end of each year for 20 years. Each payment will repay an equal amount of principal.

Immediately after the loan was made, ABC Loans sold the right to receive Ani's payments to XYZ investing for a price that will earn XYZ an annual effective return of 4%. Find the price paid by XYZ.

Example 3.25

Larry and David each borrow 4800.

Larry is charged an annual effective rate of 6%. He repays his debt by making level annual payments at the end of each year for 12 years.

David is charged an annual effective rate of i. He repays his debt by making payments at the end of each year for 12 years. David's payments each repay an equal amount of principal.

The total of Larry's payments is equal to the total of David's payments. Find i.