CHAPTER 4 – Bonds

4.1 BOND VALUATION

A bond is a mechanism for borrowing money that is often used by federal and local governments, as well as by large corporations. The purchaser of a bond (i.e. the lender) receives regular interest payments (called coupons) for a fixed period of years. On the maturity date of the bond, the lender receives a payment (called the redemption amount) that is generally equal to the original purchase price of the bond.

Before introducing the notation and formulas used for bond valuation, let's consider an introductory example.

Example 4.1

Garret pays 1000 to purchase a bond. The bond pays semiannual coupons (interest payments) at a nominal rate of 6% convertible semiannually and is redeemed for 1000 at the end of 10 years.

Two years later, immediately after receiving the fourth coupon, Garret decides to sell the bond to Stella. This gives Stella the right to collect the remaining coupon payments, as well as the redemption amount of 1000. Interest rates have dropped over the course of the two years, and the price that Garret charges Stella will allow her to earn a nominal semiannual rate of 4% on her investment.

a) What price did Stella pay for the bond?
b) What nominal semiannual rate did Garret actually earn during the two year period?

Bond Terminology and Notation
We will use the following notation and terminology when working with bonds.

- P is the current price of the bond.
- F is the face amount, or par value. This amount is used to determine the size of the coupons.
- C is the redemption amount. Unless otherwise stated, bonds are redeemable at par, meaning that $C=F$.
- n is the number of remaining coupon payments.
- r is the effective coupon rate per payment period. The size of the coupon is thus Fr.
- g is a special coupon rate that is occasionally used in formulas when $C \neq F$. It is defined by $Cg = Fr$.
- i is the effective interest rate per payment period earned by the purchaser of the bond.

We make the following comments regarding the terms introduced above:

- **Coupon Rate vs Interest Rate.** Bond problems involve two different rates. The coupon rate is used ONLY to determine the size of coupon payments. The interest rate is the rate that is actually used to price the bond once the coupons have been determined. The reason for the different rates is that interest rates can change over time, but the coupons are set when the bond is initially issued. See Example 4.1.
- **Price, Par Value, and Redemption Amount.** The price P is what is paid for the bond. The par value F is used only for determining the coupon size. The redemption amount C is what is actually repaid at redemption. Bonds we consider will generally be redeemable at par $(C=F)$. For a bond redeemable at par, if $r = i$ then the price is equal to the redemption amount and $P=C=F$.
- **Special Coupon Rate.** Assume that a bond is not redeemable at par. If we replace the coupon rate with the special coupon rate, then we can now consider the bond to be redeemable at par.

Bond Valuation Formulas

We will use the following two formulas to price bonds:

- Basic Bond Valuation Formula: $P = F\,r\,a_{\overline{n}|i} + C\,v^n$
- Premium/Discount Formula: $P = C + (F\,r - C\,i)\,a_{\overline{n}|i}$

The first formula can be derived using basic annuity concepts. The premium/discount formula can be derived from the basic formula using algebraic methods. The basic formula will be the one that we use for most bond problems. However, some problems are more easily solved using the premium discount formula. The advantage of the P/D formula is that the variable n only appears in one place in the formula.

Note on rates: It is important to note that the rates r and i used in the bond formulas above are effective rates for the stated coupon period. Most bonds pay semiannual coupons. In that case, r and i are both **effective semiannual** rates.

We now consider two basic examples utilizing the bond valuation formulas.

Example 4.2

A 15-year 1000 par value bond yields 4% convertible semiannually. Coupons are paid semiannually. The bond is redeemable at par.

a) Find the purchase price of the bond if it pays coupons at 3% convertible semiannually.
b) Find the purchase price of the bond if it pays coupons at 5% convertible semiannually.

Example 4.3

A 2000 par bond pays coupons semiannually at 5% per annum and is redeemable at par after 10 years. The price of the bond is 1900. Find the nominal semiannual yield rate of the bond.

A zero coupon bond is a bond that pays no coupons. The price of a zero coupon bond is simply the present value of its redemption amount. Example 4.4 involves such a bond.

Example 4.4

Hailey buys three bonds. Each bond has a par value of 1000, matures in n years, and is priced to yield an annual effective rate of i. You are given:

i) The first bond is a zero coupon bond and has a price of 402.78.
ii) The second bond pays 8% annual coupons and has a price of 1167.22.
iii) The third bond pays 5% annual coupons and has a price of P.

Find P.

Example 4.5

Elliot purchases a 20-year, 1000 par bond. The bond pays semiannual coupons at a rate of 6% convertible semiannually and is priced to yield an annual effective rate of i. As the coupon payments arrive, Elliot reinvests them into an account earning 5% convertible semiannually. At the end of the 20 year period, Elliot's overall effective annual yield is 7%. Calculate i.

Example 4.6

Darlene pays P for a 15 year, 2000 par bond paying semiannual coupons at 6%. After 6 years, immediately after the 12th coupon, Darlene sells the bond to Angela at a price that yields Darlene a rate of 8% convertible semiannually and yields Angela a rate of 7% convertible semiannually. Find P.

4.2 PREMIUM AND DISCOUNT

As we have seen in previous examples, if $r > i$ for a par-value bond, then $P > C$. Similar, if $r < i$ for a par-value bond, then $P < C$. We will now introduce terminology to refer to these two situations.

- **Premium.** If $P > C$ for a bond, then the bond is said to be purchased at a *premium*. For par-value bonds, this occurs when $r > i$. In general, a bond is at a premium if $g > i$.
- **Discount.** If $P < C$ for a bond, then the bond is said to be purchased at a *discount*. For par-value bonds, this occurs when $r < i$. In general, a bond is at a premium if $g < i$.

Amount of Premium or Discount

- For a bond purchased at a premium, the value $P - C$ is referred to as the **amount of premium**.
- For a bond purchased at a discount, the value $C - P$ is referred to as the **amount of discount**.

If using the P/D formula $P = C + (Fr - Ci)a_{\overline{n}|i}$ to price a bond, the quantity $(Fr - Ci)a_{\overline{n}|i}$ will be positive if the bond is sold at a premium and negative if the bond is sold at a discount. In either case, $|Fr - Ci|a_{\overline{n}|i}$ will be equal to the amount of premium or discount.

Example 4.7

A 15-year 1000 par value bond yields 4% convertible semiannually. Coupons are paid semiannually The bond is redeemable at par. Find the amount of premium or discount if:

a) The bond pays coupons at 3% convertible semiannually.
b) The bond it pays coupons at 5% convertible semiannually.

Book Value of a Bond

The book value of the bond is the current price of the bond, if it were to be resold at the same yield rate as when it was purchased. Alternately, one can think of the book value of a bond as being the current outstanding loan balance. We will use B_t to refer to the book value immediately after payment number *t*. In the next section, we will discuss the book value of a bond at times between two coupon payments. For now, however, we are only interested in B_t at times immediately after a coupon payment has been made. Notice the following:

- $B_0 = P$ since P is the amount that is initially borrowed by the bond issuer.
- $B_n = C$ since C is the amount that is ultimately repaid by the bond issuer.
- At any other time, B_t is equal to the PV of all future payments. It follows that $B_t = Fr\,a_{\overline{n-t}|i} + Cv^{n-t}$.

Noting that $B_0 = P$ and $B_n = C$, we can make the following observations about the book value of a bond:

- **Premium.** If $P > C$, then the book value decreases over time. This is because the coupon payments exceed the interest accumulated by the outstanding balance. The excess is applied to the principal.
- **Discount.** If $P < C$, then the book value increases over time. This is because the coupon payments are smaller than the interest accumulated by the outstanding balance. The deficit is added to the principal.
- **Neither.** If $P = C$, then $B_t = P = C$ after every coupon payment. In this case, the coupon payments are exactly equal to the interest accumulated on the loan.

We illustrate these concepts with the following example.

Example 4.8

Construct amortization tables for bonds with the following parameters:

a) $C = 1000$, $n = 3$, $r = 10\%$, $i = 8.0578\%$, $P = 1050$
b) $C = 1000$, $n = 3$, $r = 10\%$, $i = 12.0848\%$, $P = 950$
c) $C = 1000$, $n = 3$, $r = 10\%$, $i = 10\%$, $P = 1000$

Write-Down of Premium and Write-Up of Discount

Let I_t be the interest accumulated on the outstanding balance of the bond at time *t*. Let $P_t = |I_t - Fr|$. Then P_t is the absolute value of the difference between the accumulated interest and the coupon payment.

- For a bond purchased at a **premium**:
  - P_t is subtracted from the book value. That is, $B_t = B_{t-1} - P_t$.
  - P_t is referred to as the **write-down of premium**, or the **amount for amortization of premium**.
  - It can be shown that $P_t = (Fr - Ci)v^{n-t+1}$.
- For a bond purchased at a **discount**:
  - P_t is added to the outstanding balance. That is, $B_t = B_{t-1} + P_t$.
  - P_t is referred to as the **write-up of discount**, or the **amount for accumulation of discount**.
  - It can be shown that $P_t = (Ci - Fr)v^{n-t+1}$.
- In either case, we have that $P_t = |Fr - Ci|v^{n-t+1}$.
- It is important to note that the values P_t form a geometric sequence with common ratio $(1+i)$.

Example 4.9

A 15 year, 5000 par value bond pays semiannual coupons at 8% and is purchased to yield 6% convertible semiannually.

a) Find P_{10}, the write-down of premium in the 10th coupon payment.
b) Find I_{10}, the interest portion of the 10th coupon payment.
c) Find B_{10}, the book value after the 10th coupon payment.

Premium/Discount Formula

As the name implies, the P/D bond valuation formula can be useful in problems involving premium or discount.

Example 4.10

A 1000 par, 12-year bond pays semiannual coupons. The bond is purchased at a discount to yield 8% compounded semiannually. The amount for accumulation of discount in the 18th coupon is 15. Find the amount of discount in the original purchase price.

Example 4.11

A 1000 par, 12-year bond pays 5.5% annual coupons and is purchased at a discount to yield 8.5% annually. The write-up in value during the first year is 8.48. Find the purchase price of the bond.

Comparing Book Values

Thinking of book value for a bond as the outstanding balance of the loan allows us to construct a relationship between the book values of a bond at two different times. Consider two times t_1 and t_2 such that $t_1 < t_2$. Let $k = t_2 - t_1$. Then $B_{t_1} = Fr\,a_{\overline{k}|i} + B_{t_2}v^k$.

Example 4.12

Piper purchases an *n*-year 1000 par bond. The bond pays annual coupons at a rate of 6%. The book value of the bond at the end of 3 years is 1161.09. The book value of the bond at the end of 5 years is 1145.24. Find the price of the bond.

4.3 PRICES BETWEEN COUPON DATES

Bonds are regularly bought and sold on the market. It is certainly not always true that the time of purchase for a bond will fall immediately after a coupon has been paid. In this section, we will consider two methods for pricing bonds at times that fall between two coupon payments.

Full Price

The *full price* (also called the *dirty price*, *flat price*, or *price-plus-accrued*) is equal to the book value of the bond immediately after the most recent coupon, accumulated at interest. Thus, the full price of a bond k coupon periods (with $0 < k < 1$) after the *n*th coupon is given by the formula $B_{t+k} = (1+i)^k B_t$.

Market Price

The *market price* (also called *clean price*, or simply *price*) is the full price minus the portion of the next coupon that has so far been "accumulated". For instance, at time $t+k$, the book value is B_{t+k} and the amount of the coupon that has been "accumulated" would be $k\,F\,r$. This, the market price at this time is $B_{t+k} - k\,F\,r$.

Full Price vs. Market Price

It is important to note that the full price is the actual price of the bond at time $t+k$. If a bond is sold at time $t+k$, then B_{t+k} is the price paid for the bond. The market price is essentially an estimation. To understand the purpose of considering the market price, notice that the full price does not change continuously over time. Assume that a coupon payment has just been made. The dirty price will continuously increase as interest is accumulated over the course of the next coupon period. However, once the coupon payment has been made, the dirty price will drop by the amount of the coupon, causing a discontinuity in the price. The market price provides a continuous (although technically less correct) estimation of the bond price. Some financial institutions report bond prices using the full price and some use the market price.

Summary of Formulas

- Full Price: $B_{t+k} = (1+i)^k B_t$
- Market Price: $B_{t+k} - k\,F\,r$

Example 4.13

A 5-year, 1000 par bond pays semiannual coupons at a rate of 10%. The bond is purchased to yield 7% compounded semi-annually.

a) Find the dirty price of the bond 15 months after its purchase, assuming the same yield.
b) Find the clean price of the bond 15 months after its purchase, assuming the same yield.

Example 4.14

A 2000 par value 12 year bond pays 6% semiannual coupons. The yield rate is 8% convertible semiannually.

a) Find the dirty price of the bond 7.2 years after its purchase, assuming the same yield.
b) Find the clean price of the bond 7.2 years after its purchase, assuming the same yield.

4.4 CALLABLE BONDS

For the types of bonds we have been discussing, the maturity date is set when the bond is created. That is to say that the date on which the borrower repays the redemption amount is determined from the outset and the borrower is not given the option of early repayment. A **callable bond** is one in which the borrower is given the option to repay the redemption amount of the bond prior to the originally stated maturity date. Typically, a callable bond will have a specific range of dates leading up to the maturity date during which early repayment is an option. Some callable bonds will also have a variable redemption amount that depends on the date on which the bond is called.

A crucial thing to understand about callable bonds is that the stated yield rate assumes that the bond will be held until maturity. If the bond is called early, that will almost certainly have an effect on the yield rate realized by the lender. Whether the yield increases or decreases depends on whether the bond was sold at a premium or discount. This concept is illustrated in the following two examples.

Example 4.15 A 10-year 1000 par bond delivers 10% annual coupons. The bond was purchased for 1050 and can be called at any point after 5 years.

a) Determine the yield on the bond if it is not called early.
b) Determine the yield if the bond is called immediately after the 8th coupon is paid.

Example 4.16 A 10-year 1000 par bond delivers 10% annual coupons. The bond was purchased for 950 and can be called at any point after 5 years.

a) Determine the yield on the bond if it is not called early.
b) Determine the yield if the bond is called immediately after the 8th coupon is paid.

Effects of Early Redemption

- For a bond sold at a premium, earlier redemption dates will result in a lower yield. This can be remembered using the mnemonic device PEW, which stands for: "Premium: Earlier is Worse".
- For a bond sold at a discount, earlier redemption dates will result in a higher yield. This can be remembered using the mnemonic device DEB, which stands for: "Discount: Earlier is Better".

Example 4.17 Peter pays 1514.52 for a 24-year par value bond paying coupons semiannually at a rate of 6%. The bond can be called at par on any coupon date starting at the end of year 17. The price paid by Peter guarantees him a yield of at least 5% compounded semiannually.

a) Calculate the par value of this bond.
b) Calculate the highest yield that Peter might earn on this bond.

Example 4.18 Bruce purchases a 16-year 10,000 par bond paying 5% semiannual coupons. The bond is callable at par on any coupon date beginning at the end of year 9. The price paid by price guarantees him a yield of at least 7% convertible semiannually.

a) Find the price paid by Bruce.
b) Calculate the highest yield that Bruce might earn on this bond

4.5 SPOT RATES AND FORWARD RATES

Up to this point, when calculating present values we have generally been provided with one rate of interest that we would use to calculate the present value of any payment, regardless of when it occurred. In practice, however, the yield rate that you can get on an investment such as a bond tends to depend on the length of time until the bond matures. Longer term bonds usually (but not always) have higher yields than bonds with shorter terms. Two reasons why a bond purchaser might demand a higher return for a long-term bond are given below.

1. Assume a company issues 5-year bonds and 30-year bonds. There is a greater risk of the company defaulting on the 30-year bond than the 5-year bond. An investor in the 30-year bond would likely want a higher return to compensate for the additional risk.
2. An individual investing in a long-term bond will have their money tied up in the bond for an extended period of time, and will be forgoing the ability to invest their money in other opportunities that might come along later.

Spot Rates

A **spot rate** is a yield rate for a zero-coupon bond. More specifically, the n-year spot rate, denoted by s_n, is the annual effective yield for n-year zero-coupon bonds currently on the market. Thus, the price today for an n-year zero-coupon bond paying 1 can be calculated using $P = 1 / (1 + s_n)^n$. A table or graph that reports spot rates for a range of years is called a **yield curve**.

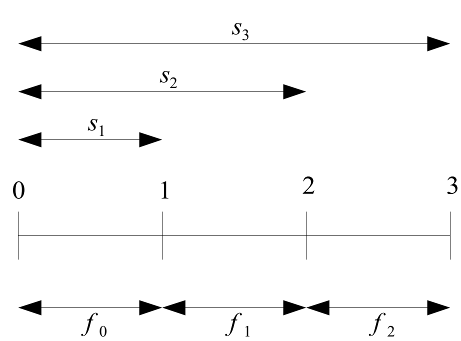
Example 4.19 Consider the yield curve provided below.

Years until Maturity	1	2	3	4
Yield on a Zero-Coupon Bond	5.00%	6.00%	6.75%	7.25%

a) Find the prices of zero-coupon bonds maturing for 100 in 1, 2, 3, and 4 years.
b) Find the price of a four-year 1000-par bond paying annual coupons of 50.
c) Find the yield rate for the bond whose price was calculated in Part b.

Forward Rates

Recall that a spot rate is an effective rate for a multiple year period beginning today. That said, any yield curve stated in terms of spot rates will imply specific annual effective rates of interest for any one year period covered by the yield curve. Such an effective annual rate is called a **forward rate**. The forward rate f_n is the effective rate for year $n+1$, as implied by the given yield curve. The diagram to the right illustrates this concept. The relationships between spot and forward rates are provided below.



- $1+f_n = \dfrac{(1+s_{n+1})^{n+1}}{(1+s_n)^n}$
- $(1+s_n)^n = (1+f_0)\cdot(1+f_1)\cdot\ldots\cdot(1+f_{n-1})$

Example 4.20 Find the forward rates implied by the yield curve given below.

Years until Maturity	1	2	3	4
Yield on a Zero-Coupon Bond	5.00%	6.00%	6.75%	7.25%

Locking In Forward Rates

Interest rates change frequently. Knowing a yield curve today doesn't tell you what rates will be in the future. When we say that the forward rate f_n is the effective rate for year $n+1$, we don't mean that this is what the one-year rate *will* be when year $n+1$ arrives. We mean that, according to current spot rates, this is the rate that should be used for year $n+1$ when doing calculations today. That said, it is possible to set up transactions that will guarantee you an effective rate of f_n during year $n+1$. The steps in doing so are described below.

1. Borrow 1, agreeing to repay $(1+s_n)^n$ in n years.
2. Immediately reinvest the 1 that was borrowed into an ($n+1$)-year bond paying $(1+s_{n+1})^{n+1}$.
3. There is no net investment at time 0.
4. A liability of $(1+s_n)^n$ will occur at time n, and an asset of $(1+s_{n+1})^{n+1}$ will be paid at time $n+1$.
5. The effective yield during year $n+1$ is given by $\frac{(1+s_{n+1})^{n+1}}{(1+s_n)^n} - 1 = f_n$.

Example 4.21 Current prices for 1000 par zero-coupon bonds are given below.

Years until Maturity	1	2	3
Price	952.38	X	843.64

The one-year forward rate for year 2 is 6%. Find X.

Example 4.22 The current five-year spot rate is 8%. The forward rate for year 2 is 6%. The current spot rate for a three-year bond purchased at time 2 is 10%. Find the one-year spot rate.

Example 4.23 Current spot rates for n-year zero coupon bonds are provided below.

Years until Maturity	1	2	3	4
Yield on a Zero-Coupon Bond	3.00%	4.00%	4.50%	5.00%

A five-year 1000 par bond pays annual coupons of 6% and is priced according to current spot rates. The bond price results in an annual effective yield of 6.5%. Find the five-year spot rate.