CHAPTER 5 – Yield Rates

5.1 DETERMINANTS OF INTERESTS RATES

Supply and Demand of Money

Consumption refers to the expenditure of money to purchase goods or services. **Interest** can be viewed as either the compensation for delaying consumption or the cost for advancing consumption, depending on whether it is viewed from the perspective of the lender or the borrower.

We begin by discussing the meaning of interest when viewed from the differing perspectives of the lender and the borrower.

- **Lender Perspective:** From the lender's perspective, interest is the compensation received for delaying consumption. As interest rates increase, it becomes more enticing to delay consumption, and more individuals with money are willing to lend that money. The availability of money to borrow will increase.
- **Borrower Perspective:** From the borrower's perspective, interest is the cost associated with making a purchase when the money for that purchase is not currently available. Thus, it can be viewed as the cost for advancing consumption. As interest rates increases, it becomes easier to

Effects of Interest Rates on Borrowing and Lending

As interest rates increase, it becomes more enticing to delay consumption by lending money. As a result, the availability of money to borrow will increase with the interest rate. On the other hand, larger interest rates represent higher costs for borrowing money. So, as rates increase, the number of people willing to borrow will decrease. If rates are very low, then there will be many people interested in borrowing, but few people willing to lend. When rates are very high, there will be many people willing to lend, but few people willing to borrow the money that is available.

It stands to reason that within a particular lending market, there will be a certain interest rate at which the supply of money to borrow will be equal to demand for borrowing. Economic theory suggests that the current interest rate within that market should tend toward this **equilibrium rate** at which the supply and demand for money are equal. There are many factors that effect interest rate levels, but this supply and demand perspective provides us with a simplified view of how interest rates are determined within a lending market.

Components of Interest Rates

In this section, we will see how interest rates can be decomposed into several different component rates, which are called **interest rate determinants**. Throughout this section, we will represent interest rates, and their determinants, as continuously compounded rates. The reason for this is that the relationship between an interest rate and its determinants is more easily expressed in terms of continuously compounded rates than with annual effective rates.

We will discuss four interest rate determinants: the **real risk-free rate**, the **maturity risk**, the **default risk premium**, and the **inflation premium**. The current interest rate, which we will denote as R in this section, will be considered to be the sum of these four components.

Real Risk-Free Rate

The **real risk-free rate**, denoted by *r*, represents the true increase in purchasing power that the lender would expect to see as the result of the loan, in the absence of risk factors such as maturity risk, default risk, and inflation (all of which will be discussed later). The real-risk free rate acts as a "base rate" used when calculating the true interest rate, *R*.

Maturity Risk

As a general principle, lenders usually demand higher rates when making long term loans than they do when making short term loans. We will discuss the reasons for this momentarily, but the basic idea is that longer term loans carry a higher level of uncertainty for the lender, and so lenders insist on receiving a higher rate to compensate them for this additional risk. We will use r_M to denote the rate determinant associated with maturity risk, also called the **maturity risk premium.** When maturity risk is the only type of risk being considered by the lender, the interest rate is given by $R = r + r_M$.

Example 5.1

Suppose the three year yield curve is given by the forward rates $f_0 = 4\%$, $f_1 = 5\%$, and $f_2 = 7\%$, expressed as continuously compounded rates of interest. Assume that the real risk-free rate *r* for short-term loans is equal to f_0 . Determine the maturity risk premium for a three year loan.

There are several theories related to the general principle that lenders require higher compensation for making long-term loans. Four of the most important theories are discussed below.

- **Market Segmentation Theory.** The market segmentation theory assumes that individual lenders and borrowers typically enter the market with a preferred loan term already in mind. As a result, the lending market naturally segments itself based upon the loan terms desired by the individuals within the market. For simplicity, assume that loans are only available in 5, 10, and 20 year terms. Then the market segmentation theory predicts that there will be three distinct markets: One for 5-year loans, one for 10-year loans, and one for 20-year loans. Each of these markets will have its own supply and demand curves, and could thus each have its own distinct interest rates. The market segmentation theory allows for the possibility for rates to be different for loans of different terms, but does not predict whether long-term rates will be higher or lower than short-term rates.

- **Liquidity Preference Theory.** When a lender makes a loan, they relinquish access to those funds during the term of the loan. This represents a loss of opportunity, since the lender will not be able to use those funds to take advantage of a better investment opportunity, should one come along prior to the maturity date of the original loan. The liquidity preference theory, also called the opportunity cost theory, asserts that lenders naturally prefer shorter-term loans to maintain flexibility in how they make their investments. As a result, lenders would thus demand a higher rate when committing their funds to a long-term loan.

- **Preferred Habitat Theory.** This theory builds onto the market segmentation theory by also asserting that individuals might be compelled to take a loan that is not of their preferred term, if the compensation for doing so was sufficiently high. For example, a borrower seeking a long-term loan, might be tempted to take a short-term loan instead, if the rate in the short-term market was sufficiently low.

- **Expectations Theory.** This theory states that long-term rates provide information about expected short-term rates in the future. For example, given the three-year spot rate s_3 and four-year spot rate s_4 , one can calculate the expected one year forward rate f_3 inferred by these spot rates.

Default Risk Premium

When a borrower fails to repay a loan, they are said to **default** on the loan. Every loan carries with it some risk of default. Lenders will generally attempt to assess the magnitude of this risk, and adjust the interest rate in order to compensate for the risk that the borrower will default. The **default risk premium**, denoted by s, is the continuously compounded risk premium charged by the lender in order to offset the default risk for the loan. If default risk is the only type of risk being accounted for in the interest rate, then we have $R = r + s$.

The size of the risk premium s depends on whether or not the lender can expect to recover any part of the loan amount in the case of a default. The next example considers an example where no money is recovered when the borrower defaults.

Example 5.2

A lending organization groups its borrowers into three risk categories: low-risk, medium risk, and high-risk. Based on past information, the lender expects that for five-year loans, 2% of all low-risk clients will default , 5% of all medium-risk clients will default, and 10% of all high-risk clients will default. The lender does not expect to receive a partial payment when a borrower defaults.

The lender would like to achieve an expected continuously compounded return of $r = 4\%$ on five-year loans made to borrowers in each risk group. Determine the rate that the lender should charge to each group, as well as the default risk premium for each group.

In the next example, we will calculate the default risk premium under the assumption that the lender is able to recover a portion of the loan amount in the case of a default.

Example 5.3

A lender is making a three-year loan to a borrower. Based on the borrower's financial history, the lender assesses that there is an 8% chance that the borrower will default on the loan. The lender collects collateral equal to 20% of the repayment amount of the loan. The lender will claim this collateral in the case that the borrower defaults. Assuming that the lender would like to see an expected return of 6% on the loan, determine the true rate that should be charged, as well as the default risk premium.

Inflation

Prices of goods and services change over time, with a tendency to increase. This effect is know as **inflation**. The rate of increase is called the **inflation rate**. In the United States, the inflation rate is typically measured by one of two indexes, the **Consumer Price Index (CPI)** or the **Producer Price Index (PPI)**. The details on how these indexes are calculated differ, but the idea is similar. Each index tracks the price of a specific goods and services over time, and calculates a weighted average of current prices to determine the current value of the index. The rate of change in either of these indexes serves as an estimate for the inflation rate.

Example 5.4

The value of the CPI two years ago today was 237.42. The value of the index today is 246.52. Use these values to estimate the continuously compounded rate of inflation over the last two years.

A consequence of inflation is that the inherent value, or purchasing power, of a single unit of currency tends to decrease over time as prices increase. When a lender makes a loan, they need to account for inflation when setting their desired interest rate. If a lender sets an interest rate of 2%, but prices increase by a rate of 3% during the term of the loan, the the amount received by the lender at maturity will be numerically greater than the original loan amount, but will have a smaller amount of purchasing power.

The inflation rate is never known in advance. If it were, then the lender could simply account for inflation by adding the inflation rate to the desired real risk-free rate for the loan. The following example illustrates this idea.

Example 5.5

Consider a four-year loan of 1000. Suppose that there is no risk of default for the loan.

a) Assuming that there is no inflation, the lender requires a return of 6%. Calculate the repayment amount required by the lender.

b) Assume that the continuously compounded inflation rate for the next four years is known to be 1.5%. Calculate the amount of money that would carry the same amount of purchasing power as the repayment amount calculated in Part (a).

c) To account for the effects of inflation, the lender demands an interest rate that would yield a repayment amount equal to the amount found in Part (b). Determine the interest rate charges on the loan.

As mentioned above, it is unrealistic to assume that the interest rate is known in advance. One way in which a lender can account for the effects of inflation is to include **inflation protection** in the loan. In an inflation-protected loan, a desired interest rate will be set, from which a base repayment amount can be calculated. The actual amount repaid by the borrower when the loan matures will be this base amount adjusted according to the actual inflation observed during the term of the loan. In other words, the final repayment amount will be set so as to yield a rate of $R = r + i_a$, where r is the desired real risk-free rate, and i_a is the actual observed rate of inflation during the loan term. Note that this does not account for default risk. To additionally account for default risk, one would need to add in a risk premium, yielding $R = r + s + i_a$.

Example 5.6

Consider a 6-year loan of 1000 with inflation-protection. The loan agreement specifies a continuously compounded interest rate of 4%, with an inflation adjustment determined by the percentage increase in the CPI during the term of the loan. Assume that the CPI is equal to 232.10 when the loan is entered into, and is equal to 258.57 when the loan is repaid. Determine the amount repaid by the borrower.

While insurance protection provides a good solution for the lender, it might not be desirable to the buyer, who would likely prefer to know the exact amount that they would eventually be required to repay when entering into the loan. If inflation protect is not an option, then the lender might simply add on an estimate of what they expect the inflation rate to be based on recent history. Denote this estimate by i. If we assume that the rate includes premiums for inflation and default risk, then the interest rate would be given by $R = r + s + i$

Example 5.7

Consider a four-year loan of 1000. The lender desires a real risk-free rate of 5%. The lender estimates that there is a 10% chance of the borrower default. If the borrower does default, the lender anticipates that he will be able to recover 25% of the repayment value of the loan. The lender estimates that the continuously compounded rate of inflation will be 1.75% over the next four years. Determine the interest rate that the lender should charge.

5.2 DISCOUNTED CASH FLOW ANALYSIS

Net Present Value

- Assume a company is considering a project that is expected to require several investments, but is also expected to generate several payments.
- We will call the investments required *cash outflows*, or *liabilities*. If an outflow occurs at time *t*, we will denote it by L_t .
- The income generated for the company will be called *cashed inflows*, or *assets*. If an inflow occurs at time *t*, we will denote it by A_t .
- We generically refer to the collection of cash inflows and cash outflows as *cash flows*. If a cash flow occurs at time *t*, we often denote it by CF_t . To distinguish between liabilities and assets when using this notation, liabilities are set to be negative.
- Let PV_L be the present value of all liabilities associated with the project and let PV_A be the present value of all assets.
- The *net present value (NPV)* of the project is given by $NPV = PV_A - PV_L$.
- If $NPV > 0$, then the project is a good venture of the company. If $NPV < 0$, then the project will generate net losses for the company.
- When calculating NPV, an interest rate must by chosen. The rate used is called the *cost of capital* or the *interest preference rate*. It is generally the interest rate at which a company is able to borrow and lend money.

Example 5.8

Consider the following two cash streams:

i) $A_0 = 720,\ L_1 = 1700,\ A_2 = 100$

ii) $A_0 = 235,\ L_2 = 250$

Compare the NPV of these two cash streams using $i = 4\%$ as well as $i = 8\%$.

Internal Rate of Return

- Given a series of cash flows, the *internal rate of return (IRR)* of the cash flows is the interest rate at which the NPV of the cash flows is zero.
- The IRR of a series of cash flows is not necessarily unique.
- It is often impractical to calculate the IRR without using a financial calculator or a computer.

Calculating NPV and IRR with the BA II Plus

The BA II Plus can be used to calculate NPV and IRR for an irregular series of cash flows. The process can be somewhat complicated, however. A few examples are provided below.

1. Assuming that $i = 4\%$, show that the NPV of the following series of cash flows is 116.9698:
   $CF_0 = -400,\ CF_1 = 0,\ CF_2 = 200,\ CF_3 = 0,\ CF_4 = 100,\ CF_5 = 300$.
   - [CF] [2ND] [CE/C] 400 [+/-] [ENTER] [↓] 0 [ENTER] [↓] [↓] 200 [ENTER] [↓] [↓] 0 [ENTER] [↓] [↓] 100 [ENTER] [↓] [↓] 300 [ENTER] [↓] [↓] [NPV] 4 [ENTER] [↓] [CPT]

2. Change the interest rate in the previous problem to $i = 10\%$ to get a NPV of 19.8670.
   - [↑] 10 [ENTER] [↓] [CPT]

3. Assuming that $i = 6\%$, show that the NPV of the following series of cash flows is 113.3178:
   $CF_0 = 0,\ CF_1 = -200,\ CF_2 = 100,\ CF_3 = 100,\ CF_4 = 100,\ CF_5 = -300,\ CF_6 = 200,\ CF_7 = 200$.
   - [CF] [2ND] [CE/C] 0 [ENTER] [↓] 200 [+/-] [ENTER] [↓] [↓] 100 [ENTER] [↓] 3 [ENTER] [↓] 300 [+/-] [ENTER] [↓] [↓] 200 [ENTER] [↓] 2 [ENTER] [NPV] 6 [ENTER] [↓] [CPT]

4. Show that the IRR for the following series of cash flows is 7.9388%:
   $CF_0 = -600,\ CF_1 = 0,\ CF_2 = 0,\ CF_3 = 0,\ CF_4 = 200,\ CF_5 = 200,\ CF_6 = 500$.
   - [CF] [2ND] [CE/C] 600 [+/-] [ENTER] [↓] 0 [ENTER] [↓] 3 [ENTER] [↓] 200 [ENTER] [↓] 2 [ENTER] [↓] 500 [ENTER] [↓] [↓] [IRR] [CPT]

5.3 DOLLAR-WEIGHTED AND TIME-WEIGHTED RETURNS

Suppose you have an investment portfolio whose rate of return varies over time. Suppose also that you make occasional withdrawals from, or deposits into the portfolio. In this section we will consider two different methods of measuring the performance of such a portfolio: The dollar-weighted return, and the time-weighted return.

Dollar-Weighted Return (DWR)

As discussed in Section 5.1, the IRR is the interest rate at which the net present value of all of the transactions is 0. This could be restated by saying that the NPV of the deposits is equal to the NPV of the withdrawals. These NPV expressions can be large-degree polynomials, and thus calculating the IRR can be impractical without a computer or financial calculator. The dollar-weighted return (DWR) is a simple interest approximation of the IRR. The equations required to solve the DWR are linear in i, and thus much easier to solve.

The IRR (and hence DWR) can be heavily affected by the transactions made for the portfolio. For instance, assume that over the course of a year, a specific portfolio has a significant "up" period followed by a significant "down" period. Suppose that two investors both invest in the same portfolio, but one investor is invested only during the up period, the the other investor is only invested during the down period. The two investors will have drastically different DWRs, even though they were invested in the same portfolio.

If we wish to measure the performance of a portfolio without considering the effects of inflows and outflows, we can use the time-weighted return.

Time-Weighted Return (TWR)

Assume we wish to measure the performance of the portfolio on its own merits, without regards to any transactions posted to the account. In this case we can use the time-weighted return (TWR). The TWR is calculated by spitting the time period of concern into intervals of constant return, and then calculating the effective return from the returns of the smaller sub-periods. Although TWR doesn't take into account the effect of any transactions to the account, these transactions often have to be considered when calculating the returns during the shorter time periods.

Example 5.9

A fund collects interest at a nominal rate of 20% convertible semi-annually for $0 \le t \le 0.5$ and at a nominal rate of 10% convertible semi-annually for $0.5 \le t \le 1$.

a) Find the TWR of this fund.
b) Assume 100 is deposited into the account at $t = 0$. Find the DWR.
c) Assume that 100 is deposited at $t=0$ and 10 is deposited at $t=1/2$. Find the DWR.
d) Assume that 10 is deposited at $t=0$ and 100 is deposited at $t=1/2$. Find the DWR.

Calculating DWR and TWR from Transaction Information

Assume we are given (or have constructed) a table such as the following one containing transaction information for a fund. We will now discuss how to use such a table to calculate the DWR and TWR.

Time	$t = t_1 = 0$	$t = t_2$	$t = t_3$	$t = t_4$	$t = t_5 = 1$
Beginning Balance	0	B_2	B_3	B_4	B_5
Transaction	T_1	T_2	T_3	T_4	T_5
Ending Balance	E_1	E_2	E_3	E_4	0

Finding DWR

- The DWR is calculated by using the middle row containing the transaction information.
- Let $\Delta_i = 1 - t_i$ be the time elapsed between $t = t_i$ and $t = 1$.
- The fund is assumed to begin and end with a 0 balance, so we can set up the following equation for DWR: $T_1(1 + i\Delta_1) + T_2(1 + i\Delta_2) + T_3(1 + i\Delta_3) + T_4(1 + i\Delta_4) + T_5 = 0$.
- Solving this equation for i yields: $DWR = \dfrac{-(T_1 + T_2 + T_3 + T_4 + T_5)}{T_1\Delta_1 + T_2\Delta_2 + T_3\Delta_3 + T_4\Delta_4}$.

Finding TWR

- Notice that the accumulation factor for the time period $[t_i,\ t_{i+1}]$ is given by $\dfrac{B_{i+i}}{E_i}$.
- It follows that: $TWR = \dfrac{B_2}{E_1} \cdot \dfrac{B_3}{E_2} \cdot \dfrac{B_4}{E_3} \cdot \dfrac{B_5}{E_4} \cdot - 1$

Example 5.10

An investment account is worth 200 at the beginning of the year. Six months later, the account is worth 220, and 120 is withdrawn. Six months after that, the account is worth 85. Find the TWR and DWR during this one year period.

Example 5.11

Assume a fund contains 100 at $t = 0$.
At $t = 1/4$, the fund is worth 120, and 110 is withdrawn.
At $t = 1/2$, the fund is worth 5, and 55 is deposited.
At $t = 1$, the balance of the fund is 75.
Calculated the TWR and DWR.

Example 5.12

On January 1, Perry deposits 150 into an investment fund.
On April 1, the balance of the account is X , and W is withdrawn.
On December 31, the balance of the fund is 140.
The DWR over the 1-year period is 15.69%, and the TWR over the same period is 14.87%.
Find X .

5.4 PORTFOLIO AND INVESTMENT YEAR METHODS

Assume that several people form an investment group, with new members joining periodically. The group makes new investments every year. The effective interest rate earned by any one investment may fluctuate from year to year, and may be different from investments purchased during other years. This section discusses two methods of determining how returns should be distributed to the members.

Portfolio Method

This method ignores when members joined the investment group. At the end of any year, the total return earned on all investments is divided among the members proportionally based on the amount they had invested at the beginning of the year.

Investment Year Method (IYM)

In this method, individual returns are based on the year in which the person joined the group. In any given year, individual contributions will earn returns at different rates, depending upon when the contributions were made.

Let's consider a simple example that will hopefully explain the motivation behind these different methods.

Example 5.13

Mike and Mark start an investment group at the beginning of 2010. They each contribute 1000 and they decide to invest the combined amount of 2000 into Fund A. Over the course of 2010, Fund A earns a return of 6%. The total return is 120, which Mike and Mark split evenly. After they each take their returns of 60, the account still contains 2000 invested in Fund A.

At the beginning of 2011, Judy decides to join the investment group and contributes 1000. Mike and Mark also each contribute an additional 1000. The group decides to invest the 3000 of new money in Fund B. This leaves 2000 in Fund A and 3000 in Fund B.

Over the course of 2011, Fund A earns only a 4% return, while Fund B earns 8%. The total return was $0.04(2000) + 0.08(3000) = 320$. The question now is how to split this return between the three individuals.

Method 1. The total amount invested into the fund is 5000. Mike and Mark each contributed 40% of this amount, whereas Judy contributed only 20% of the total. If we use these percentages to allocate the returns, then Mike and Mark will each get 128 and Judy will receive 64. Under this method, all 3 individuals will earn a 6.4% return during 2011.

Method 2. Judy might argue that her money was only invested in Fund B, and thus she should earn the full 8% on her investment, or 80. That would leave 240 to be split evenly between Mike and Mark, who would each earn 120. Notice that 120 is also equal to the a 4% return on the 1000 each of the two has invested in Fund A, plus a 8% return on the 1000 that they have invested in Fund B. Under this method, Mike and Mark would each earn a total return of 6%, whereas Judy earns 8%.

What would be deemed fair in the previous example probably depends on the perspectives of those involved. Notice that had Fund A done better than Fund B during 2011, Judy would have preferred Method 1.

Reading IYM and Portfolio Rates from a Table

Investment year and portfolio rates are generally reported by using a table, such as the one below. The following comments will explain how to use this table.

- The portfolio rates for a given year are reported in the last two columns. For instance, under the portfolio method, all members earned a 6.2% return during 2011.
- The other entries provide the investment year rates, with each row representing one particular investment year. For instance, someone investing in 2010 would earn 6.4% during 2010, 6.0% during 2011, and 5.7% during 2012.
- For simplicity, it is usually the case that investments are folded into a common portfolio rate once they reach a certain age (three years, in the case of this table).
- As an example, during year 2013, contributions made in 2010 would get folded into the portfolio rate and would thus earn 5.6%. It turns out that all older contributions would also earn this rate of 5.6% in 2013. To see that this is true, notice that a contribution made in 2008 would earn 5.6% in 2008, 6.4% in 2009, 6.3% in 2010. It would then get folded into a portfolio rate making 6.2% in 2011, 5.8% in 2012, and 5.6% in 2013.

Calendar Year of Original Investment	**Investment Year Rates (in %)**			**Portfolio Rates (in %)**	**Calendar Year of Portfolio Rate**
y	i_1^y	i_2^y	i_3^y	i^{y+3}	
2006	5.6	5.8	5.2	5.5	2009
2007	6.0	5.4	6.2	5.8	2010
2008	5.6	6.4	6.3	6.2	2011
2009	6.2	6.5	5.6	5.8	2012
2010	6.4	6.0	5.7	5.6	2013
2011	5.8	5.6	5.4	5.0	2014
2012	5.4	5.6	4.4		
2013	5.2	4.6			
2014	4.8				

Example 5.14 Assume that 1000 is invested in 2009. Find the accumulated value at the end of 2014 using each of the following methods:

a) The portfolio method.
b) The investment year method.
c) Assume that the money is withdrawn at the end of each year, and reinvested at the new money rate.

Example 5.15 Quentin invests 100 at the beginning of each of the years 2010, 2011, 2012, 2013, and 2014. Using the investment year method, determine the accumulated value of Quentin's account at the end of 2014.