CHAPTER 6 – Immunization

6.1 DURATION

Assume we have initiated a project that will yield several cash inflows as well as several cash outflows at various times. Suppose we value the project using an effective annual rate of interest i. For the project to be profitable at this rate, the present value of the cash inflows and outflows must be positive. Interest rates can change over time, resulting in changes in the present value of the cash flows. If interest rates change enough, it is possible that a previously profitable venture will no longer be profitable. In the next few sections, we will study how changing interest rates affect present values.

Price Sensitivity

Assume a bond is priced using an effective yield of i. Now imagine that the next day, the yield rate has changed to $i + \Delta i$, thus changing the value of the bond. The percentage change in the price of the bond resulting from this change in the rate is called the **price sensitivity** of the bond. The price sensitivity of the bond depends strongly on the term of the bond, as we will see in the next example.

Example 6.1

Complete the following problems.

a) Find the prices of 10 and 20 year zero-coupon 1000 par bonds. Assume $i = 10\%$.
b) Find the percentage change in the prices of these bonds if the rate changes to $i = 9.8\%$.

In the previous example, the price sensitivity of the 20 year bond is roughly twice that of the 10 year bond. That is no coincidence. For zero-coupon bonds, the price sensitivity for a given change in i is approximately proportional to the time until maturity of the bond. The situation for a series of multiple cash flows is a bit more complicated to explain, and requires the introduction of the concept of "duration".

Macaulay Duration

The **Macaulay duration** (or simply **duration**) of a series of cash flows is the time-weighted average of the present values of all of the cash flows. Formulas for the Macaulay duration are given below.

- The Macaulay duration of a general series of cash flows is given by $MacD = \dfrac{\sum (t \cdot v^t \cdot CF_t)}{\sum (v^t \cdot CF_t)} = \dfrac{\sum (t \cdot v^t \cdot CF_t)}{P}$.
- The Macaulay duration of a single cash flow occurring at $t = n$ is $MacD = n$.

Example 6.2

Find the Macaulay duration of a 3-year 100-par bond paying annual coupons of 10% and yielding 8%.

Macaulay Duration and Price Sensitivity

It can be shown that the Macaulay duration of a sequence of cash flows is equal to $-P'(\delta) / P(\delta)$, where $P(\delta)$ is the present value (or price) of the sequence as a function of the force of interest δ. Thus, we can think of $MacD$ as being equal to the price sensitivity resulting from an instantaneous change in δ. However, we are more likely to calculate prices using i than δ. This observation leads us to the definition of "modified duration".

Modified Duration

We define the **modified duration** of a series of cash flows, denoted by $ModD$, to be equal to the ratio $-P'(i) / P(i)$. It can be shown that $ModD = \frac{\sum (t \cdot v^{t+1} \cdot CF_t)}{\sum (v^t \cdot CF_t)}$, or equivalently $ModD = v \cdot MacD$.

Summary of Duration Formulas

We summarize the formulas for Macaulay duration and modified duration below.

- $MacD = \frac{\sum (t \cdot v^t \cdot CF_t)}{\sum (v^t \cdot CF_t)} = \frac{\sum (t \cdot v^t \cdot CF_t)}{P} = -\frac{P'(\delta)}{P(\delta)} = (1+i) ModD$
- $ModD = \frac{\sum (t \cdot v^{t+1} \cdot CF_t)}{\sum (v^t \cdot CF_t)} = \frac{\sum (t \cdot v^{t+1} \cdot CF_t)}{P} = -\frac{P'(i)}{P(i)} = v \cdot MacD$

Example 6.3

Find the modified duration of a 20 year bond paying annual coupons of 50 and maturing for 1000. Assume an annual effective yield of 4%.

Approximating Change in Price

Assume the price of a series of cash flows is equal to P when valued using an effective rate of i. We wish to approximate the change in price ΔP resulting from a change of Δi in the rate. We can rewrite the expression $ModD = -P'(i) / P(i)$ as $P'(i) = -P(i) \cdot ModD$. Since $P'(i) \approx \Delta P / \Delta i$, it follows that $\Delta P \approx -ModD \cdot P \cdot \Delta i$.

Example 6.4

Assuming an annual effective interest rate of $i = 8\%$, an asset stream currently has a present value of 2500. The modified duration of the asset stream is 12.6. Approximate the change in the present value of this stream of payments if the interest rate suddenly increases to $i = 8.5\%$.

Duration of a Perpetuity

The duration of a perpetuity can be calculated in much the same way as any other stream of payments. The primary difference is that the sums involved will now be infinite sums. Consider the following example:

Example 6.5

A perpetuity makes payments of 4 at the end of each year. Assuming an annual effective interest rate of *i*, the perpetuity has a duration of 32.25. Find the price of the perpetuity.

Duration of a Portfolio

Let A and B be two series of payments and let C be a third stream that combines the payments delivered by A and B. The Macaulay duration of C is the price-weighted average of the durations of A and B. In other words:

- $MacD_C = \frac{P_A MacD_A + P_B MacD_B}{P_A + P_B}$

6.2 CONVEXITY

In Section 6.1, we defined modified duration of an asset with specified cash flows in terms of the derivative of the price of that asset with respect to the interest rate. We now define **convexity** in a similar manner, instead using the second derivative of the price with respect to the interest rate. Formulas for convexity are given as follows:

- $Conv = \dfrac{P''(i)}{P} = \dfrac{\sum\left[t\cdot(t+1)\cdot v^{t+2}\cdot CF_t\right]}{P}$

In the following two examples, the summation formula for convexity will probably be the most useful.

Example 6.6 An asset will make payments of 500, 200, and 300 at the end of years 3, 5, and 6, respectively. Assuming an effective annual rate of 6%, calculate the convexity of this asset.

Example 6.7 A 4-year bond pays annual coupons of 6% and has an annual effective yield of 8%. Find the modified duration and the convexity of this bond.

In the next example, the derivative definition of convexity will be the easiest to apply.

Example 6.8 A perpetuity makes payments at the end of each year. The first payment is equal to 5, and subsequent payments increase by 5 per year. Find the modified duration and convexity of this perpetuity, assuming an annual effective yield of 4%.

Approximating Change in Price

In Section 6.1, we used the relationship between $P'(i)$ and modified duration to come up with a first-order approximation for ΔP as a function of Δi. We can use Taylor series to develop a second-order approximation by incorporating convexity into our formula. This yields the following approximation:

- $\Delta P \approx P(i)\cdot\left[-(\Delta i)ModD + \frac{1}{2}(\Delta i)^2(Conv)\right]$

Example 6.9 Assuming an annual effective interest rate of $i = 4\%$, an asset stream currently has a present value of 5000. The modified duration of the asset stream is 5 and its convexity is 40. Approximate the change in the present value of this stream of payments if the interest rate suddenly decreases to $i = 3.8\%$.

Convexity of a Portfolio

Let A and B be two series of payments and let C be a third stream that combines the payments delivered by A and B. The convexity of C is the price-weighted average of the convexities of A and B. In other words:

- $Conv_C = \dfrac{P_A Conv_A + P_B Conv_B}{P_A + P_B}$

Example 6.10 Portfolio A has a present value of 320, a duration of 8.75, and a convexity of 80. Portfolio B has a present value of 180, a duration of 12.5, and a convexity of 120. The two portfolios are combined into a single portfolio. Find the duration and convexity of the new portfolio.

6.3 IMMUNIZATION

Assume a portfolio contains several cash inflows as well as several cash outflows. The net present value of such a portfolio is obviously affected by the current effective rates. In fact, the NPV might be positive at one interest rate and negative when calculated using a different rate. The effect that changing rates have on NPV poses a risk to investors and financial institutions. Such entities often employ strategies to minimize their exposure to these interest rate risks. In this section, we will introduce three such methods: Redington immunization, full immunization, and exact matching.

Redington Immunization

A sequence of cash flows is said to be in **Redington immunization** if the following three conditions hold:

1. The PV of the assets equals the PV of the liabilities. That is, $P_A(i) = P_L(i)$.
2. The duration of the assets equals the duration of the liabilities. Equivalently, $P_A{}'(i) = P_L{}'(i)$.
3. The convexity of the assets is greater than the convexity of the liabilities. Equivalently, $P_A{}''(i) > P_L{}''(i)$.

Redington immunization protects the investor from small changes in the interest rate.

The first criteria ensures that the current NPV is zero. The second criteria guarantees that the NPV has a critical point at the current value of *i*. The third criteria results in that critical point being a local minimum for the NPV.

If a set of cash flows satisfies the first two criteria of Redington immunization, it is said to be **duration matched**.

Example 6.11

Two sets of liabilities are given below. Each set of liabilities is duration matched using 2-year and 5-year zero coupon bonds. For each set of liabilities, find the par value of the bond that need to be purchased, and then determine if Redington immunization has been achieved. Assume an annual effective yield of 5%.

a) Liability of 500 at time 1 and another liability of 300 at time 6.
b) Liability of 500 at time 3 and another liability of 300 at time 4.

Example 6.12

A company has a liability portfolio with a present value of 600, a duration of 8, and a convexity of 168. The company plans to duration match its liabilities using the following asset portfolios:

- Portfolio A, which has a duration of 10.25 and a convexity of 210
- Portfolio B, which has a duration of 6.5 and a convexity of *K*.

Find the smallest value of *K* that will achieve Redington immunization.

Full Immunization

A financial enterprise is said to be in **full immunization** if the following three conditions hold:

1. The PV of the assets equals the PV of the liabilities. That is, $P_A(i) = P_L(i)$.
2. The duration of the assets equals the duration of the liabilities. Equivalently, $P_A{}'(i) = P_L{}'(i)$.
3. There is one cash inflow before and after each cash outflow. That is, there no two consecutive cashflows that are both liabilities.

Full immunization protects the investor from all changes in the interest rate.

Example 6.13

A liability of 1000 to be repaid at time 6 is fully immunized using an 8-year zero coupon bond and an n-year zero coupon bond. The par value of the 8-year bond is 648.96. The current annual effective interest rate is 4%. Find the par value of the n-year bond.

Example 6.14

BusinessCorp has a liability of 500 due n years from now. They fully immunize the liability by investing in a zero coupon bond that matures for 267 in n – 2 years, as well as a zero coupon bond maturing for 238.2 in n + t years. The current annual effective rate of interest is 6%. Find t.

Exact Matching (Dedication)

Another immunization strategy is to match every liability with an asset to be delivered at the same time and in the same amount as the liability so that there is a net cash flow of 0 at all times. This strategy is called **exact matching** or **dedication**.

Example 6.15

A company has liabilities of 3500 at the end of year 1, 5000 at the end of year 2, and 6500 at the end of year 3. The company exactly matches the liabilities by investing in the following bonds:

i) A one-year zero coupon bond with a yield of 2.5%.
ii) A two-year zero coupon bond with a yield of 3%.
iii) A three-year bond paying annual coupons of 5% and priced to yield 4%.

Find the total amount invested in the three bonds.

Example 6.16

Skyler has liabilities of 2000 due at the end of each of the next three years. She uses dedication to match the liabilities by investing in the following bonds.

i) A one-year bond paying 4% annual coupons.
ii) A two-year bond paying 5% annual coupons.
iii) A three-year bond paying annual coupons at a rate of r.

At an annual effective yield of 6%, the price of the one-year bond was 1716.98. Find r.