CHAPTER 10 – Binomial Trees

10.1 INTRODUCTION TO BINOMIAL TREES

The payoff for a put or a call is a function of the price of the underlying stock on the expiration date of the option. In order to appropriately price options, we first need construct a probabilistic model for the price of the stock. We can then use this model to determine the probability that a particular option will be exercised, as well as the expected payoff of the option. We can then use this information to price the option.

There are two commonly used stock price models: the binomial tree model, as well as the Black-Scholes framework. In this chapter, we will student the binomial tree model. The basic "one-period binomial tree" model that we start with is a very simplistic model, but we will see later that it serves as the building block for more complicated and more realistic stock models.

One-Period Binomial Trees

A one-step binomial tree model is described as follows.

- Let S be the current price of the stock. This will sometimes be denoted by S_0 .
- The model covers a specific length of time. The period length is denoted by h, and is measured in years.
- We assume that there are only two possible values of the stock at time h. Either the price of the stock will increase to a value of S_u , or it will decrease to a value of S_d .
- The values S_u and S_d are sometimes stated explicitly, but are often provided as multiples of S. If the multipliers u and d are provided, then $S_u = S\cdot u$ and $S_d = S\cdot d$.
- The probability of an up-move is denoted by p . The probability of a down-move is equal to $q = 1 - p$.

Expected Stock Price and Expected Annual Return

Given a one-period binomial tree, we may calculate the following.

- The **expected price** of the stock after 1 period is $E\left[S_h\right] = p\cdot S\cdot u + q\cdot S\cdot d$.
- The **capital gains rate** g is the continuously compounded annual rate of growth that would cause the initial stock price of S to grow to the expected value $E\left[S_h\right]$. In other words, $S\,e^{gh} = E\left[S_h\right]$. It follows that $g = \frac{1}{h}\cdot\ln\left(\frac{E\left[S_h\right]}{S}\right) = \frac{\ln\left(p\cdot u + q\cdot d\right)}{h}$.
- The **continuously compounded expected annual rate of dividend growth** for the stock is denoted by δ . We will consider only non-dividend paying stocks in this section.
- The **continuously compounded expected annual rate of return** α for the stock is equal to the capital gains rate plus the continuously compounded expected rate of dividend grown, δ . That is, $\alpha = g + \delta$.

Notice that the expected rate of return α is the expected return on investment for someone purchasing the stock. They would expect return of g due to price changes, as well as an additional return of δ due to dividend growth. This gives an total return of $\alpha = g + \delta$. For a risk-averse investor to be interested in a particular stock, they would require α to be larger than the risk-free rate r .

Example 10.1

The price of a non-dividend-paying stock currently worth 50 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 8 months. The probability of an up-move is $p = 0.65$.

a) Determine the expected price of the stock after 8 months.

b) Find the continuously compounded expected annual return for the stock over the 8 month period.

Example 10.2

The price of a non-dividend-paying stock currently worth 100 is modeled by a one-period binomial tree with $u = 1.15$ and $d = 0.9$. The period for the tree is 9 months. The continuously compounded expected annual return for the stock is $\alpha = 8\%$. Determine the probability of an up-move.

Expected Payoff and Return for Options

Given a binomial tree model for a stock, we can use the model to determine the expected payoff as well as the expected return for a call or put on that stock expiring at the end of the period. The details are provided below.

- Consider a *K*-strike European call and a *K*-strike European put on a stock, both expiring at the end of one period. We will denote the payoff of the call at the up-node by C_u, and the payoff of the call at the down-node by C_d. Similarly, we will denote the payoff of the put at the up and down nodes by P_u and P_d, respectively. Formulas for these quantities are given by:
  - $C_u = \max\left[0,\, S_u - K\right]$ and $C_d = \max\left[0,\, S_d - K\right]$
  - $P_u = \max\left[0,\, K - S_u\right]$ and $C_d = \max\left[0,\, K - S_d\right]$
- The expected payoffs of the options are given by $E\left[\text{Call } PO\right] = p\,C_u + q\,C_d$ and $E\left[\text{Put } PO\right] = p\,P_u + q\,P_d$.
- The continuously compounded expected annual return for a particular option is denoted by γ. It is defined by $\text{Premium} \cdot e^{\gamma h} = E\left[\text{Option } PO\right]$, or $\gamma = \dfrac{1}{h} \ln\left(\dfrac{E\left[\text{Option } PO\right]}{\text{Premium}}\right)$.

Example 10.3

The price of a non-dividend-paying stock currently worth 50 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 8 months. The probability of an up-move is $p = 0.65$. An 8-month 112-strike call has a premium of 4.42. The continuously compounded risk-free rate is 4%.

a) Determine the expected call payoff and the expected return for the call.

b) Determine the expected put payoff and the expected return for the put.

Example 10.4

The price of a non-dividend-paying stock currently worth 120 is modeled by a one-period binomial tree $u = 1.1$ and $d = 0.9$. The period for the tree is 6 months. A 6-month 125-strike put has a premium of 6.1915 and an expected yield of 8.4911%. Find the expected return for the stock.

10.2 REPLICATING PORTFOLIOS

Given a binomial tree for a stock, it is not difficult to calculate the expected payoff for an option on that stock. The premium for the option should then be the present value of this expected payoff. The issue we face is that the rate that we should use to discount the expected payoff is γ, the expected yield of the option. If we don't already know the premium, we don't have a method of calculating γ. This requires us to develop alternate methods for pricing options. We will learn two methods: replicating portfolios and risk-neutral pricing. We will cover replicating portfolios in this section, and risk-neutral pricing in the next.

Replicating Portfolios

The method of replicating portfolios allows us to price options on a stock modeled by a binomial tree without using any probabilistic concepts. In this method, we will construct a portfolio consisting of some shares of the underlying asset, as well as some amount of borrowing of lending. The portfolio will be built so that the payoffs at time h are exactly the same as the option in consideration at both the upper and lower nodes. Since those are the only two "possible" payoffs for the option in the binomial tree model, we conclude that the price of the option must be the same as the price of our replicating portfolio.

The process of pricing an option using a replicating portfolio is outlined below.

- Assume a stock is modeled using a 1 period binomial tree with S, r, δ, h, u, and d given.
- We construct a portfolio by buying Δ shares of the stock, and investing (lending) B in risk-free bonds.
- The cost of our replicating portfolio is $\Delta S + B$.
- The portfolios value at $t = h$ is $\Delta e^{\delta h} Su + Be^{rh}$ at the up node and $\Delta e^{\delta h} Sd + Be^{rh}$ at the down node.
- Let V_u and V_d be the payoffs for the option being priced at the upper and lower nodes respectively.
- For either a call or a put, we set $\Delta e^{\delta h} Su + Be^{rh} = V_u$ and $\Delta e^{\delta h} Sd + Be^{rh} = V_d$. Solving this system yields $\Delta = \left(\frac{V_u - V_d}{Su - Sd}\right)e^{-\delta h}$ and $B = \left(\frac{uV_d - dV_u}{u - d}\right)e^{-rh}$.
- The price of the option is then $V = \Delta S + B$.
- Note that for a call, $\Delta \geq 0$ and $B \leq 0$. In contrast, for a put we have $\Delta \leq 0$ and $B \geq 0$.
- Let Δ_C be the number of shares in the replicating portfolio for a K-strike call and let Δ_P be the number of shares in the replicating portfolio for a K-strike put. A consequence of the previous derivations is that $\Delta_C - \Delta_P = e^{-\delta h}$.

We begin by looking at a few examples involving non-dividend-paying stocks.

Example 10.5

The price of a non-dividend-paying stock currently worth 130 is modeled by a one-period binomial tree $u = 1.2$ and $d = 0.8$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4.5%.

a) Calculate the premium of a one-year 128-strike call on the stock.

b) Calculate the premium of a one-year 128-strike put on the stock.

Example 10.6

The price of a non-dividend-paying stock currently worth 120 is modeled by a one-period binomial tree $u = 1.3$ and $d = 0.85$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4%. Find the strike price of a one-year call option whose replicating portfolio contains 0.5926 shares of the stock.

We will now consider an example involving dividend-paying stocks.

Example 10.7

The price of a stock currently worth 100 is modeled by a one-period binomial tree $u = 1.3$ and $d = 0.8$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 5%.

c) Calculate the premium of a one-year 98-strike call on the stock.

d) Calculate the premium of a one-year 98-strike put on the stock.

Example 10.8

The price of a stock is modeled using a one-period binomial tree with a period of six months. The difference between the price of the stock at the upper and lower nodes is 48. The difference between the payoffs of a six-month K-strike call at the upper and lower nodes is 31. The stock pays continuous dividends at a rate of 3%. Find the number of shares in the replicating portfolio for a six-month K-strike put on the stock.

10.3 RISK NEUTRAL PRICING

The method of risk neutral pricing provides an alternative to replicating portfolios for pricing options on stocks whose prices are modeled using binomial trees. The two methods produce the same results, but each have their own advantages. For some applications, the value of delta found using the replicating portfolio method might be of interest in its own rate. An advantage of risk neutral pricing is that it tends to be easier to apply when working with a multi-period binomial tree.

Risk Neutral Pricing Method

When using risk neutral pricing, we assume that we are in a "risk-neutral" world where every investment is expected to grow at the risk-free rate. The details of the method are explained below.

- Assume that the underlying stock is modeled by a one period binomial tree with parameters S , u , d , δ , and h . Also assume that the risk-free rate r is given.
- To use the risk neutral model, we do not need to know the value of p . Recall that if we did know p , then we could be able to calculate the expected price of the stock $E[S_h] = p \cdot S \cdot u + q \cdot S \cdot d$ **.** We could subsequently calculate the capital gains rate g using $S e^{gh} = E[S_h]$, and the expected yield $\alpha = g + \delta$.
- When using risk neutral pricing, we assume that the expected yield is equal to r and then calculate the **risk neutral probability** p^* consistent with this return. This amounts to solving for p^* in the equation $p^* \cdot S \cdot u + (1 - p^*) \cdot S \cdot d = S e^{(r-\delta)h}$.
- The quantity $E^*[S_h] = p^* \cdot S \cdot u + (1 - p^*) \cdot S \cdot d$ is called the **risk neutral expected value of the stock**.
- Solving for p^* gives the formula $p^* = \dfrac{e^{(r-\delta)h} - d}{u - d}$.
- Assume now that we wish to price an option that has values of V_u and V_d at the up and downs. The **risk neutral expected payoff** of the option is given by $E^*[PO] = p^* V_u + (1 - p^*) V_d$.
- The premium is then obtained by discounting the risk neutral expected payoff using the risk-free rate. That is: $\text{Premium} = \left[p^* V_u + (1 - p^*) V_d \right] e^{-rh}$.

Example 10.9

The price of a stock currently worth 100 is modeled by a one-period binomial tree $u = 1.3$ and $d = 0.8$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 5%.

a) Use risk neutral pricing to price a one-year 98-strike call on the stock.

b) Use risk neutral pricing to price a one-year 98-strike put on the stock.

The problem we just solved is identical to the one presented in Example 10.7. Compare the results to verify that the two methods do in fact generate the same prices.

Example 10.10

The price of a stock currently worth 140 is modeled by a one-period binomial tree $u = 1.25$ and $d = 0.85$. The probability of an up-move is $p = 0.4$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 4.5%. Find the expected yield for a one-year 135-strike European put on the stock.

10.4 MULTI-PERIOD BINOMIAL TREES

As mentioned previously, one-period binomial trees are not a particularly realistic model for stock prices. We can obtain more realistic models by expanding upon the idea and considering multi-period binomial trees. The details of the multi-period binomial tree model are explained below.

- Assume that the length of time covered by the model is T years, and that this interval of time is split into n periods of length h.
- The initial stock price is S. Prices at later nodes are denoted using subscripts indicating the number of up and down moves required to reach that node.
- The probability of an up-move for any given period is p. In the case of an up-move, the price is multiplied by a factor of u. The multiplier for a down-move is d.

Pricing European Options Using Multi-Period Binomial Trees

We will explain how to price a European Option using a two-period binomial tree. The process for using a binomial tree with more than two periods is a natural extension to this method.

1. Assume that we are pricing a $2h$-year K-strike European option.
2. Denote the payoffs of the option at each of the three terminal nodes by V_{uu}, V_{ud}, and V_{dd}.
3. Use the payoffs V_{uu} and V_{ud} to calculate the price of a h-year K-strike option of the same type, sold at time h, assuming that an up-move occurred during the first period. Denote this quantity by V_u.
4. Use the payoffs V_{ud} and V_{dd} to calculate the price of a h-year K-strike option of the same type, sold at time h, assuming that a down-move occurred during the first period. Denote this quantity by V_d.
5. Use the values V_u and V_d to determine the price of the option, V.

An alternate (and equivalent) method would be to calculate the risk-neutral expected payoff at time $2h$ using $E^*\left[S_{2h}\right] = \left(p^*\right)^2 S_{uu} + 2\,p^* q^* S_{ud} + \left(q^*\right)^2 S_{dd}$, and then discount to time 0 using the risk-free rate: $V = E^*\left[S_{2h}\right] e^{-2rh}$.

Example 10.11

The price of a stock currently worth 160 is modeled by a two-period binomial tree $u = 1.2$ and $d = 0.8$. Each period is 6 months. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 6%.

a) Calculate the premium for a one-year 185-strike European call on the stock.

b) Calculate the premium for a one-year 185-strike European put on the stock.

Pricing American Options Using Multi-Period Binomial Trees

Recall that American options and European options differ in that European options can only be exercised when the option expires, whereas an American option can be exercised at any point prior to the expiration date for the option. We can use multi-period binomial trees to price American options by making a small adjustment to the process using for European options. We explain the process for a two period binomial tree below.

1. Assume that we are pricing a $2h$-year K-strike American option. We also assume that the option in question can only be exercised at the end of an h-year period.

2. Denote the payoffs of the option at each of the three terminal nodes by V_{uu}, V_{ud}, and V_{dd}.

3. We now calculate V_u. The process is more complicated than with European options.
   a) Use the payoffs V_{uu} and V_{ud} to find the price of a h-year K-strike option of the same type, sold at time h, assuming that an up-move occurred during the first period. Denote this quantity by MV_u.
   b) Find the payoff for the option at the up-node if it were exercised early. Denote this by PO_u.
   c) Let $V_u = \max\left[MV_u, PO_u\right]$.

4. We now calculate V_u.
   a) Use the payoffs V_{ud} and V_{dd} to find the price of a h-year K-strike option of the same type, sold at time h, assuming that a down-move occurred during the first period. Denote this quantity by MV_d.
   b) Find the payoff for the option at the down-node if it were exercised early. Denote this by PO_d.
   c) Let $V_d = \max\left[MV_d, PO_d\right]$.

5. Use the values V_u and V_d to determine the price of the option, V.

Example 10.12

The price of a stock currently worth 160 is modeled by a two-period binomial tree $u = 1.2$ and $d = 0.8$. Each period is 6 months. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 6%.

a) Calculate the premium for a one-year 185-strike American call on the stock.

b) Calculate the premium for a one-year 185-strike American put on the stock.