CHAPTER 1 – The Measurement of Interest

1.1 ACCUMULATION FUNCTIONS AND THE TIME VALUE OF MONEY

Time Value of Money

Everything we will learn in this course is based upon the concept of the time value of money. **Time value of money** refers to the idea that receiving a certain amount of money today is worth more than receiving the same amount of money at some future time. To illustrate the intuition behind this idea, consider the following scenario.

> Assume that you win a small lottery. You are given the choice of two payments: \$100 right now, or \$$K$ to be paid one year from now. Let's consider under what conditions you would be enticed to take the later payment.
>
> If $K < 100$, then you would certainly take the earlier payment. It would not make sense to wait one year to receive less money. Even if $K = 100$, you should almost surely take the immediate payment. The payments are in the same amount, so there is no incentive for you to delay receipt of the payment. The *amounts* of the immediate payment and the delayed payment are the same, but you would likely attribute some additional worth to being able to receive the money immediately. Thus, we can say that the *time value* of a payment of \$100 to be received immediately is greater than the *time value* of the payment of \$100 delivered one year from now.
>
> It is clear that for you to consider taking the later payment, it would need to be in an amount larger than \$100. Take a moment to try to consider how large K would need to be for you to choose to take the later payment rather than receiving \$100 immediately.
>
> For the sake of discussion, let's assume that you have decided that you would take the earlier payment if $K < 120$, and you would take the later payment if $K > 120$. Let's also assume that if $K = 120$, then you would be equally happy with either payment. If this is the case, then a payment of \$100 today would have the same inherent worth to you as a payment of \$120 paid one year from now. We could say that the two payments have the same *time value* to you, even if they are in different amounts.
>
> It is entirely possible that someone else might have a different perspective on the relative value of these payments. Someone who has little immediate need for additional money might be more willing to take a later payment, and would thus require a smaller incentive to delay the payment. Such an individual might consider \$100 paid today to have the same time value as \$105 paid one year from now. This person would gladly trade \$100 today in order to receive \$110 one year from now, whereas you would refuse such an offer.

Although the exact way in which time effects the perceived value of a payment is subjective and can vary from one person to the next, it is hopefully clear that the time at which a payment occurs does effect the inherent value of that payment.

When two entities enter into an agreement that involves an exchange of money at different times, they will generally need to decide ahead of time on a method for determining the time value of money so that they can compare the value of payments occurring at different times. This goal can be achieved through the use of accumulation and amount functions, which we will introduce soon.

Interest

The most common application of the time value of money is interest. Let's assume that an amount P is borrowed, under the condition that the borrower will repay the loan at some point in the future. The concept of the time value of money dictates that for this arrangement to be fair to the lender, the amount repaid must be greater than the initial amount borrowed. Let's denote the amount repaid as $P+I$. The quantity P is the original loan amount, which is also referred to as the **principal** of the loan. The amount I is the **interest** on the loan, and can be thought of as a fee that the borrower pays to the lender to compensate the lender for temporarily giving up access to their money.

Loans are a type of investment. The lender can be thought to be investing the loan principle with the expectation of receiving a larger amount in return at some point in the future. Interest is also paid on other types of investments, such as bonds, which we will discuss in detail in later sections.

Accumulation and Amount Functions

In this class, we will use the concepts of accumulation and amount functions to track the growth rate and value of an investment that is earning interest. The definitions of these types of functions are provided in the box below.

Accumulation and Amount Functions

Assume that an amount of principal P is invested at time $t = 0$ in an account that earns interest.

- The **amount function** for the account is a function $A(t)$ that provides the value, or **balance**, of the account at time t. It follows that $A(0) = P$.
- The **accumulation function** for the account is a function $a(t)$ that provides the multiple by which the account has grown during the first t years.
- It follows from the definitions that $A(t)$ and $a(t)$ are related by $A(t) = P\,a(t)$.

There are two criteria that a function must satisfy to be a valid accumulation function. These are that $a(0) = 1$ and $a(t) > 0$ for all $t \geq 0$.

It is important to remember that an accumulation function $a(t)$ is required to satisfy the property $a(0) = 1$. We will see problems later in which we are required to find accumulation functions satisfying certain criteria. This property can help make this task easier, and also provides a way of double-checking our answers.

Example 1.1

An account earns interest according to the accumulation function $a(t) = 1 + 0.01\,t^2$.
Assume 100 is invested into the account at time 0.

a) Find the value of the account at the end of 2 years.
b) Find the value of the account at the end of 5 years

Example 1.2

A loan accumulates interest according to an accumulation function of the form $a(t) = e^{kt}$. The loan is to be repaid with a single payment of 2642.07 at the end of 10 years. An amount of 2078.33 was owed at the end of 6 years. Find the original loan amount.

We will see a wide variety of accumulation functions in this course, and most new concepts that we introduce early on will be explained in terms of general accumulation functions. That said, there are two specific forms of accumulation functions that will be of particular importance to us. Those are the accumulation functions associated with **simple interest** and **compound interest**. In simple interest, the account balance increases linearly over time, as opposed to compound interest where the account balance increases exponentially.

Simple Interest

Simple interest is a form of interest accumulation in which the value of the investment increases linearly over time. There are infinitely many simple interest accumulation functions, but they all take a similar form. Specific simple interest accumulation functions are defined by the choice for the value of a special constant i, called the simple interest rate.

Simple Interest

- Let i be a constant, which we will call the **annual simple interest rate**.
- The simple interest accumulation function defined by i is given by $a(t) = 1 + it$.

Since the accumulation function for simple interest is a linear function of t, the net growth in the account over any two time periods of equal length will be exactly the same.

Example 1.3

A loan of 200 is charged simple interest at an annual rate of 10%.

a) Find the amount owed at the end of 5 years.
b) Find the increase in the amount owed during year 5.
c) Find the increase in the amount owed during year 6.

Simple interest is generally applied when working with short-term loans or investments whose duration is less than one or two years. For longer-term investments, it is more standard to use compound interest, which we will discuss next.

Compound Interest (Annual Compounding)

Compound interest refers to any form of interest in which the accumulation is represented by an exponential function. Nearly all of the problems you will encounter in this course will use accumulation functions associated with compound interest, and you will be introduced to many alternate forms for such accumulation functions. We will begin by considering the most basic form of accumulation function associated with compound interest, interest with annual compounding.

Compound Interest (Annual Compounding)

- Let i be be a constant, which we will call the **annual effective interest rate**.
- The compound interest accumulation function defined by i is given by $a(t) = (1+i)^t$.

Example 1.4

A loan of 200 is charged compound interest at an annual effective rate of 10%.

a) Find the amount owed at the end of 5 years.
b) Find the increase in the amount owed during year 5.
c) Find the increase in the amount owed during year 6.

As mentioned previously, most of the problems encountered in this course will deal with compound interest. The instruction to use compound interest is sometimes related by explicitly stating that an investment earns compound interest. In other cases, however, this information is conveyed in the way that the interest rate is described. Any time that you see an interest rate described using some variation of the words "annual" and "effective", you should assume that you to being asked to use compound interest.

We will now give several examples of sentences that can be used to indicate the use of compound interest accumulation functions. The following statements all mean exactly the same thing.

- A loan is charged compound interest at an annual effective rate of 6%.
- A loan is charged interest at an annual effective rate of 6%.
- A loan is charged interest at a rate of 6% annual effective.
- A loan is charged interest at an effective rate of 6% per annum.
- A loan is charged interest at a rate of 6% compounded annually.

Example 1.5

Erlich simultaneously makes investments into two different funds: Fund A, and Fund B.

- Fund A earns interest at a rate of 4% annual effective.
- Fund B earns interest at a 6% annual effective rate.
- The amount invested in Fund A was twice the amount invested in Fund B.
- After 6 years, the two accounts are collectively worth $20,000.

Determine the combined value of the funds 10 years after the initial investments.

Example 1.6

Lucy and Patty each invest 200. Lucy's investment earns compound interest and Patty's investment earns simple interest. At the end of 2 years, the two investments are of equal value. At the end of 4 years, the value of Lucy's investment is 1.05 times that of Patty's investment. Find the value of Patty's investment after 5 years.

The motivation behind the formula for annual compounding is the assumption that interest is applied to the account at the end of each year. The interest accumulated each year is added to the current balance, and will itself earn interest during the subsequent years. As a result, the value of the account will increase by a factor of $1+i$ each year. After t years, the value will have increased by a factor of $a(t) = (1+i)^t$.

This interpretation implies that the balance is constant throughout the year, only increasing at the end of the year. This would technically result in a piece-wise defined accumulation function with jump discontinuities at the end of each year when the compounding occurs. To avoid complications stemming from working with discontinuous functions, we will assume that the balance is continuously increasing and that the the accumulation function $a(t) = (1+i)^t$ is valid for all times t.

Example 1.7

Beth deposits 200 into an account that earns a 4% annual effective rate of interest. Jerry deposits 300 into an account that earns interest at an effective rate of 8% per annum. How many years pass until the two accounts have the same balance? Round your answer to 2 decimal places.

Compound Interest (Continuous Compounding)

As mentioned earlier, there are other types of accumulation functions that can be used to represent compound interest. The choice of which form to use depends upon how the rate you are provided has been described. We already know that when we are told that the rate is an "annual effective rate", then you would use the accumulation function $a(t) = (1 + i)^t$, where i denotes the rate. If, however, you are told that the rate is **compounded continuously**, then you would use an accumulation function of the form described below.

> **Compound Interest (Continuous Compounding)**
> - Let δ be be a constant, which we will call either the **continuously compounded rate of interest** or the **force of interest**.
> - The compound interest accumulation function defined by δ is given by $a(t) = e^{\delta t}$.

Although we will tend to use δ to denote the continuously compounded rate of interest, it is also quite common to see *r* used to denote this rate.

Example 1.8 A fund earns interest at a continuously compounded rate of 12%. How long does it take for the value of the fund to double?

We have introduced two quite different-looking functions for compound interest, $a(t) = (1 + i)^t$ and $a(t) = e^{\delta t}$. The forms for these accumulation functions are not as different as they might seem at first glance. They are, in fact, simply different expressions of the same idea. Any function of the form $a(t) = (1 + i)^t$ can also be written in the form $a(t) = e^{\delta t}$, although the values of i and δ will differ slightly.

It might seem redundant to introduce two different ways of expressing the same concept, but both of these forms are useful. Each form has situations for which it is more convenient to use than the alternate form.

Example 1.9 Assume that $\delta = 8\%$ and $(1 + i)^t = e^{\delta t}$. Find i.

The rates δ and i are said to be equivalent rates since the accumulation functions associated with them are the same, and thus they describe the same rates of growth (although in slightly different ways). We will discuss equivalent rates in more detail in Section 1.5.

Compound Interest: Varying Annual Rates

It is not uncommon for an account earning compound interest to earn different effective rates during different years. The process for dealing with such scenarios is explained below.

- Assume an account earns an effective annual rate of i_1 during the first year, i_2 during the second year, and so on, eventually earning a rate of i_t during year *t*.
- The accumulation function for the account is given by $a(t) = (1 + i_1)\cdot(1 + i_2)\cdot\ldots\cdot(1 + i_t)$.

Example 1.10 Joe deposits 100 at time 0. His account earns 5% in year 1, 7% in year 2, and 4% in year 3. Find the value of the account at the end of the third year.

Amount of Interest Earned

Assume that a certain amout of principal is invested into an account at time 0, and no other payments are made into the account. The amount of interest earned by the account during a specific time period is equal to the increase in the value of the account during that time period. More precisely, the amount of interest earned during the period between times t_1 and t_2 is equal to $A(t_2) - A(t_1)$.

Example 1.11 A loan of 750 is charged interest according to the accumulation function $a(t) = 1 + 0.08t^2$. Determine the amount of interest accumulated between the end of the 16th and 24th months.

We are often asked to determine the amount of interest earned on an investment during a specific one year period. Let I_t denote the interest accumulated during year *t*, where *t* is a whole number. Since year *t* starts at time $t-1$ and ends at time t, we get that $I_t = A(t) - A(t-1)$.

We can apply the formula $I_t = A(t) - A(t-1)$ when working with any sort of amount function. But since we will predominately work with simple and compound interest, it is useful to have special formulas for I_t to apply when we are working specifically with these types of interest. Such formulas are provided below. There derivations are not show, but they following directly from the expression $I_t = A(t) - A(t-1)$.

Amount of Interest Earned During Year *t*

Assume a deposit of *P* is made into an account earning interest according to some accumulation function. Let I_t refer to the amount of interest earned during year *t*.

- In general, we have that $I_t = A(t) - A(t-1)$.
- If the account earns simple interest at an annual rate of *i*, then $I_t = i \cdot P$.
- If the account earns compound interest at an annual effective rate of *i*, then $I_t = i \cdot A(t-1)$.

Example 1.12 Steven deposits 500 into Fund X and 500 into Fund Y at time 0. Fund X earns simple interest at an annual rate of 6%. Fund Y earns compound interest at an annual effective rate of 6%. Calculate the total amount of interest that Steven earns during the fourth year.

1.2 EFFECTIVE RATES OF INTEREST

In the previous section, we defined the concept of an "annual effective rate of interest". Such an interest rate was defined specifically for use when working with compound interest. It turns out, however, that we can generalize the term "effective rate of interest" in ways that allow it to be applied to situations involving any accumulation functions, not just those associated with compound interest. We will also discuss effective rates of interest over non-annual periods.

Annual Effective Rate For a Given Year

Assume that an investment grows according to some accumulation function $a(t)$. The **annual effective rate of interest during a given year** measures the percentage growth in an account during that specific year. We will denote the effective annual rate during year *t* by i_t. It is important to note that year *t* runs from time $t-1$ to time t. Formulas for i_t are provided below.

Annual Effective Rate During Year *t*

- Assume that an investment grows according to an accumulation function $a(t)$.
- The annual effective rate of interest during year *t* is given by: $i_t = \dfrac{A(t) - A(t-1)}{A(t-1)}$.
- It can be shown that i_t can also be calculated using the formula: $i_t = \dfrac{a(t) - a(t-1)}{a(t-1)}$.

Example 1.13 An account earns interest according to the accumulation function $a(t) = e^{0.01t^2}$.

a) Find i_6, the annual effective rate of interest during year 6.
b) Find i_7, the annual effective rate of interest during year 7.
c) Find i_8, the annual effective rate of interest during year 8.

Assume that an account earns compound interest at an annual effective rate of *i*. It can be shown that i_t is constant for such an account, and in fact $i_t = i$ for any *t*. This illustrates that the phrase "annual effective rate" is consistent between the definition presented here and the one provided for compound interest. In fact, i_t can be thought of as the rate of growth in an account over a one year period *assuming* that growth was the result of accumulation by compound interest.

Although i_t is constant in the case of compound interest, it is a decreasing function of *t* when working with simple interest. This fact is demonstrated in the following example.

Example 1.14 An account earns simple interest at an annual rate of 10%.

a) Show that $i_t = \dfrac{0.1}{0.1t + 0.9}$.
b) Find i_t during each of the first five years.
c) During what year is i_t first less than 5%?

Calculator Tip: Parts (b) and (c) in Example 1.14 can be easily solved using the "table" function of the TI-30XS calculator. I recommend reading about this function in the calculator manual. It can be very useful.

Annual Effective Rate Over a Period

Assume that an investment grows according to an accumulation function $a(t)$. The **annual effective rate of interest during a *t* year period** is defined to be the equal to the annual effective rate of compound interest that would have produced the same amount of growth during that same period of time.

> **Annual Effective Rate During a *t* Year Period**
>
> - Assume that an investment grows according to an accumulation function $a(t)$.
> - The annual effective rate of interest during the first t years can be found by solving for i in the following equation: $(1+i)^t = a(t)$.

The annual effective rate of interest during a period is found by assuming that total amount of growth seen during that period was the result of compound interest.

Example 1.15 An account earns interest according to the accumulation function $a(t) = e^{0.01t^2}$.

a) Find the annual effective rate of interest over the course of the first 4 years.
b) Find the annual effective rate of interest over the course of the first 8 years.

Example 1.16 An account earns simple interest at an annual rate of 10%.

a) Find the annual effective rate of interest during the first 5 years.
b) Find the annual effective rate of interest during the first 10 years.

Compare the results of the next example to the answer in Example 1.9.

Example 1.17 An account earns compound interest continuously at a rate of 8%.

c) Find the annual effective rate of interest during the first 5 years.
d) Find the annual effective rate of interest during the first 10 years.

The annual effective rate over a period of several years can be thought of as a sort of average of the annual effective rates for each of the years during that period. The average is not, however, a standard arithmetic average, as illustrated in the next example.

Example 1.18 An account earns 5% in year 1, 3% in year 2, and 7% in year 3. Find the annual effective rate of interest earned over the course of the three years.

1.3 PRESENT VALUE AND CURRENT VALUE

Present Value

As mentioned in Section 1.1, the inherent value of a payment depends in part on when the payment is received. If we would like to compare the value of two payments occurring at different times, then we need to select a specific time and use an accumulation function to determine the value of each of the two payments at this particular time. As we will see later, the particular time t at which we value the payments is somewhat arbitrary. However, it is often convenient to determine the time value of the two payments at time $t=0$. The time value of a payment at time $t=0$ is called the **present value (PV)** of the payment.

Present Value (PV) for General Accumulation Functions

- Assume you are provided an accumulation function $a(t)$.
- The present value of a payment of 1 occurring at time t is given by $PV = \dfrac{1}{a(t)}$.
- The present value of a payment of K occurring at time t is given by $PV = \dfrac{K}{a(t)}$.

Let PV be the the present value of a payment K due at time t. The value PV can be interpreted in several ways:

- An investment of PV at time 0 would grow to an amount of K at time t under the effects of $a(t)$.
- If a loan of PV is made at time 0 and accumulates interest according to the accumulation function $a(t)$, then K is the amount that will be owed to the lender at time t.
- According to the accumulation function $a(t)$, a payment of PV at time 0 has the same time value as a payment of K at time t.
- If an individual can readily borrow and lend money that will accumulate interest according to the accumulation function $a(t)$, then they will (in theory) place the same value on a payment of PV at time 0 as they would a payment of K at time t.

Example 1.19 Given the accumulation function $a(t) = (1 + 0.1t)^2$, find the present value of a payment of 1000 at the end of four years. In other words, determine the amount that would need to be invested at time 0 in order for the investment to grow to 1000 at the end of four years.

The process of "moving a payment back in time" in order to find its present value is called **discounting**. Recall that the process of "moving a payment forward in time" to find its future value is called **accumulating**.

Present Value for Compound Interest

Note that in order to accumulate forward t years under compound interest, then we simply multiply the original principal by a certain accumulation factor t times. The accumulation factor is equal to $(i+i)$ in the case of annual compounding and $e^{\delta t}$ for continuous compounding. It follows that if we want to discount a payment at time t back to time 0, then we simply need to divide by this factor t times. This is equivalent to multiplying by the reciprocal of the accumulation factor t times. When working with compound interest, we will refer to the reciprocal of an accumulation factor as a **present value factor**.

Present Value (PV) for Compound Interest

- Assume an account earns compound interest (either annually, or continuously). We will define the **present value factor**, denoted by v, as follows:
  - **Annual Compounding:** Let $v = \dfrac{1}{1+i}$.
  - **Continuous Compounding:** Let $v = e^{-\delta}$.
- In either case, the PV of a payment of K at time t is given by $PV = K\,v^t$.

When working with compound interest, it is important to remember the following facts:

- Multiplying the value of a payment by an accumulation factor $(1+i)$ or e^{δ} accumulates by one year, or in other words, determines the time value of the payment one year in the future. Multiplying by the accumulation factor n times carries the value of the payment forward n years.
- Multiplying the value of a payment by the present value factor v discounts the payment by one year. That is, it determines the time value of the payment at a time one year earlier. Multiplying by the present value factor n times discounts the payment by n years.

Example 1.20 Assuming an annual effective interest rate of 8%, which of the following payments has a larger present value: A payment of 120 at time 4, or a payment of 150 at time 7?

Calculator Tip: It is often the case that you will need to use the same present value factor multiple times in the same problem. For such problems, it can be convenient first calculate v, and then store it in one of the calculator registers, such as x. After doing so, if you wanted to calculate the PV of, for example, a payment of 70 at time 8, you would enter: $70\,x^8$.

Accumulating Over General Periods (The Delayed Deposit Trap)

Recall that $a(t)$ was said to provide the factor by which an account grows in the t years immediately after depositing the initial deposit or loan. This statement assumes that the initial transaction took place at time 0. In other words, $a(t)$ does not provide the accumulation factor for an arbitrary t year period, but instead one that begins at time 0.

Assume $t_1 < t_2$. Given an accumulation function $a(t)$, if we wish to accumulate a payment of P occurring at time t_1 forward to time t_2, we first discount the payment to time 0 to obtain a present value of $P / a(t_1)$. We then accumulate this PV forward to time t_2 to obtain an accumulated value of $P \cdot a(t_2) / a(t_1)$. The factor $a(t_2) / a(t_1)$ describes the growth during the period $t \in [t_1, t_2]$. We will denote this quantity by $a(t_1, t_2)$.

Accumulating Over General Periods

- Let $a(t)$ be an accumulate function and let $t_1 < t_2$.
- Define $a(t_1, t_2)$ by $a(t_1, t_2) = \dfrac{a(t_2)}{a(t_1)}$.
- Assume an account has a value of P at time t_1. The accumulated value of this account at time t_2 is given by $P \cdot a(t_1, t_2)$.

Example 1.21 The value of an account grows according to the accumulation function $a(t) = (1 + 0.1t)^2$. The account is worth 2000 at the end of the year 3. Find the value of the account at the end of year 7.

Example 1.22 A fund grows according to the accumulation function $a(t) = 1 + 0.05t^2$. A deposit of X is made into the account at time 6. Determine the number of years required for the value of the deposit to double.

Note that the situation is much simpler when working with compound interest. It can be shown that when working with compound interest, the accumulation factor during ANY t year period will be equal to $(1 + i)^t$. For all other accumulation functions, one must apply the accumulation factor $a(t_1, t_2) = a(t_2) / a(t_1)$ when accumulating over a period that does not begin at time 0.

Example 1.23 The value of an account accumulates interest an an annual effective rate of 7%. The account is worth 2000 at the end of the year 3. Find the value of the account at the end of year 7.

Present Value at a Specified Time (Current Value)

It is often necessary to discount a payment to a time other than 0. Assume that $t_1 < t_2$. To discount a payment of *K* occurring at time t_2 back to time t_1, we divide *K* by the accumulation factor $a(t_1, t_2)$ to obtain a discounted value of $K \cdot a(t_1) / a(t_2)$. This discounted value is referred to as the **present (or current) value at time t_1** of the payment, and can be interpreted as the amount that must be invested at time t_1 for the account to be worth *K* at time t_2.

Present Value at a Specific Time

- Let $a(t)$ be an accumulate function and let $t_1 < t_2$.
- Define $a(t_1, t_2)$ by $a(t_1, t_2) = \dfrac{a(t_2)}{a(t_1)}$.
- The present (or current) value at time t_1 of a payment of *K* occurring at time t_2 is given by $PV = \dfrac{K}{a(t_1, t_2)} = K \cdot \dfrac{a(t_1)}{a(t_2)}$.

Any mention of present value that does not specify a particular time will refer to the present value at time 0.

Example 1.24 Given the accumulation function $a(t) = 1 + 0.05t^2$, find the present value at time 4 of a payment of 3250 occurring at time 10.

Flexibility of $a(t_1, t_2)$

Up to this point, we have only use the function $a(t_1, t_2)$ in situations where $t_1 < t_2$.

Assume that $t_1 < t_2$. We have established the following two facts in this section:

- To accumulate a payment from time t_1 to time t_2, we multiply the payment by $a(t_2) / a(t_1)$.
- To discount a payment from time t_2 to time t_1, we multiply the payment by $a(t_1) / a(t_2)$.

In either case, we are multiplying by a factor of the form $a(t) / a(s)$, where *s* is the time of payment, and *t* is the time that we are moving the payment to. We can unify these two rules into the following rule:

- Assume a payment of *K* occurs at time *s*. The value of the payment at time *t* equal $K \cdot a(s, t) = K \cdot \dfrac{a(t)}{a(s)}$.

This new rules works whether we are accumulating or discounting.

Equivalent payments

It can be shown that if two payments have the same present value at a certain time t_1, then they will have the same present value at ALL times *t*. As a result, we consider the two payments to be **equivalent** with respect to the accumulation function being used.

1.4 PRESENT VALUE OF A SEQUENCE AND METHOD OF EQUATED TIME

Present Value of a Sequence of Payments

We will define the present value of a sequence of payments to be equal to the sum of the present values of each of the individual payments. A single payment of this amount at time 0 is considered to have the same time value as the entire sequence of payments.

Present Value of a Sequence of Payments

- Let K_1, K_2, …, K_n be the value of payments made at times t_1, t_2, …, t_n, respectively.
- The present value of the payments is given by $PV = \frac{K_1}{a(t_1)} + \frac{K_2}{a(t_2)} + ... + \frac{K_n}{a(t_n)}$.
- Assuming compound interest, then we have that $PV = K_1 v^{t_1} + K_2 v^{t_2} + ... + K_n v^{t_n}$.

Example 1.25 Assume an annual effective interest rate of 5%. Find the present value of the following sequence of payments: 100 at time 2, 300 at time 4, and 250 at time 5.

The answer in Example 1.25 is 533.40. There are several useful ways in which one can interpret this value:

- Assume that a loan of 533.40 is made at time 0 at an annual effective rate of 5%. If the borrower makes payments of 100 at time 2, 300 at time 4, and 250 at time 5, then the loan will be completely paid off after the last payment. It should be noted that this is but one of many possible ways to repay such a loan. Any sequence of payments with the same present value would represent a valid repayment plan for this loan.
- Assume that 533.40 is invested at time 0 at an annual effective rate of 5%. This deposit could fund withdrawals of 100 at time 2, 300 at time 4, and 250 at time 5. After the withdrawal at time 5, the account would be empty.
- Assuming an annual effective interest rate of 5%, a payment of 533.40 has exactly the same time value as the entire sequence of payments described in the problem.

Example 1.26 At an annual effective interest rate of *i*, the following two sets of payments described below have the same present value, *P*. Find *P*.

i) A payment of 140 at the end of year one, and a payment of 140 at the end of year four.
ii) A payment of 200 at the end of year three, and a payment of 200 at the end of year six.

We will occasionally encounter problems in which we are asked to consider two or more possible payment plans for a loan. Such a situation could occur if the borrower misses payments and thus requires a new repayment plan to be developed, or if the borrower decides to repay the loan early. The key to solving such problems is to remember that any sequence of payments whose present value is equal to the loan amount will represent a valid repayment plan.

Example 1.27 Elbert borrows 2400 at an annual effective interest rate of i. He intends to repay the loan by making a payment of 1800 after three years, and another payment of 1327.63 after six years.

Elbert pays the first payment as normal. At the end of the fourth year, Elbert decides to repay the loan by paying off the remaining balance of the loan, which is equal to K. Find K.

It is possible that when renegotiating a loan, there will be additional penalties incurred by the party wishing to renegotiate. Such penalties can have an effect on the annual effective rate ultimately realized by the loan.

Example 1.28 Assume that in the previous example, Elbert is required to pay an early payment penalty of 50 along with his second payment of K. Use the **table** function of the TI-30XS to estimate the annual effective interest rate actual paid by Elbert when the penalty is taken into account. Round your answer to three decimal places.

Example 1.29 An account earns interest at an annual effective interest rate of i. Deposits are made into the account as follows: 200 is deposited at time 0, 300 is deposited at time n, and 500 is deposited at time $2n$. The accumulated value of the fund at the end of the tenth year is 1512.12.

Given that $v^n = 0.8209$, find i.

Present Value of a Sequence of Payments at Time t_1

As with individual payments, it is possible to value a sequence of payments at any given time t_1. Doing so would produce the value of a payment that, if paid at time t_1, would have the same time value as the given sequence. There are two equivalent methods we can use to find the present (or current) value of a sequence at time at time t_1.

Present Value of a Sequence of Payments at Time t_1

The present (or current) value at time t_1 of a sequence of payments can be found by using either of the following (equivalent) methods:

1. Calculate the present value of the sequence at time 0. Accumulate the resulting present value forward to time t_1.
2. Calculate the current value at t_1 of each of the payments separately, and then sum the results. Payments occurring prior to time t_1 will need to be accumulated forward, whereas payments occurring after time t_1 will need to be discounted backward in time.

It is important to know that the two methods above produce equivalent results. However, I would generally recommend using the second method to calculate current value of a sequence.

If two sequences of payments have the same present value at time 0 (and thus at all times t) then we will consider them to be equivalent. Equivalent sequences will have the same time value. Two equivalent sequences can be viewed as both being valid repayment plans for the same loan.

Example 1.30 Dana and Ed each take out loans of the same size. The loans collect interest at an effective annual rate of 4.5%. Dana plans to pay off her loan by making a payments of 100 at the end of years n, $n+1$, $n+2$, and $n+3$. Ed agrees to repay his loan with a single payment of P at the end of year $n+1$. Find P.

Example 1.31 Morty borrows an amount of L at time 0. The loan is charged interest at an annual effective rate of 5%. According to the original terms of the loan, Morty is required to make payments of 100 at the end of each of the first four years, at which point the loan will be repaid.

Morty instead pays the loan off early by making a payment of 100 at the end of year 1, 125 at the end of year 2, and a payment of X at the end of year 3. Find X.

Method of Equated Time

Assume that n payments of $P_1, P_2, \ldots, P_n$ are made at times $t_1, t_2, \ldots, t_n$, respectively. The **method of equated time** provides an estimate $\bar{t}$ of the time at which a single payment of $P_1 + P_2 + \ldots + P_n$ has same present value as the original series of payments. The estimate $\bar{t}$ is calculated as a weighted average of the times at which the payments are made, with the weights provided by the size of the payments.

Method of Equated Time

- Assume that n payments of $P_1, P_2, \ldots, P_n$ are made at times $t_1, t_2, \ldots, t_n$, respectively.
- Let $\bar{t} = \dfrac{P_1 t_1 + P_2 t_2 + \ldots + P_n t_n}{P_1 + P_2 + \ldots + P_n}$.
- The time $\bar{t}$ provides an approximation for the time at which a payment of $P_1+P_2+\ldots+P_n$ would be equivalent to the given sequence.

Note that $\bar{t}$ is just an estimate. If we want to know the exact time t at which as single payment in an amount of $P_1 + P_2 + \ldots + P_n$ has the same present as the original series of payments, then we would need to solve the equation $\left(P_1 + P_2 + \ldots + P_n\right)v^t = P_1 v^{t_1} + P_2 v^{t_2} + \ldots + P_n v^{t_n}$.

Example 1.32 Payments of 300, 500, and 200 are made at the end of years two, four, and seven, respectively.

a) Use the method of equated time to estimate the time at which a single payment of 1000 would be equivalent to the described sequence of payments.

b) Assuming an annual effective interest rate of 6%, determine the actual time when a single payment of 1000 would be equivalent to the given sequence.

Example 1.33 John borrows P at an annual effective rate of 5%. He agrees to repay the loan by making payments of 200 at the end of year 1, 250 at the end of year 2, and X at the end of year 4.

Using the method of equated time, we determine that John could have also repaid the loan by making a payment of $450 + X$ at approximately time $t = 2.625$.

Find X.

1.5 NOMINAL RATES OF INTEREST

Effective Periodic Rates

Up to this point, we have placed an emphasis on using years as a unit of time. Annual effective rates measure the growth over the course of a year, and our time variable t is always measured in years. Although years are generally convenient to work with, there is no reason why a year must be the default unit of time. There are many types of investments that will accumulate interest every six months, as well as those that accumulate interest every month. When working with such types of investments, it may make sense to use six month or one month periods as our default unit of time. If we are using something other than years as our default unit of time, then it is usually convenient to work with non-annual effective rates that measure the growth over such a period of time.

Effective Periodic Rates for Compound Interest

- Define an ***m*-thly period** to be a length of time equal to $1/m$ years.
- There are m such periods during each year.
- Assume an account earns compound interest.
- We will define the **effective *m*-thly rate** to be the rate of growth for the account during each m-thly period. When the length of the period is clear, then we sometimes refer to this rate as the **effective periodic rate**.
- Assume that P is invested into an account earning compound interest at an m-thly effective rate of j. The value of the account after N periods is $P(1+j)^N$

As hinted at in the box above, we will generally use j (or sometimes k) to refer to an m-thly effective rate, reserving i specifically for annual effective rates.

We assign special names to m-thly rates for certain values of m:

- If $m = 2$, we will refer to an effective m-thly rate as an **effective semi-annual rate**.
- If $m = 4$, we will refer to an effective m-thly effective rate as an **effective quarterly rate**.
- If $m = 12$, we will refer to an effective m-thly effective rate as an **effective monthly rate**.

Example 1.34 Assume an account earns compound interest at an effective monthly rate of 1%. An amount of 250 is deposited into an account. Find the value of the account after 1.5 years.

Nominal Rates of Interest

It is standard practice when stating an m-thly rates to scale the rate to a full year by multiplying it by m. Such a rate is called a **nominal annual rate compounded (or convertible) *m*-thly**, and is denoted by $i^{(m)}$. As we will soon see, $i^{(m)}$ is something different from an annual effective rate. We can calculate the annual effective rate for a problem involving nominal rates of interest, but the rate i will differ slightly from the value of $i^{(m)}$.

When encountering a problem that uses a nominal rate $i^{(m)}$, first step is almost always to divide the nominal rate by m to obtain an effective periodic rate, and then move forward working on an m-thly basis.

Nominal Annual Rates of Interest

- The **nominal annual interest rate compounded (or convertible) *m*-thly** is denoted by $i^{(m)}$.
- Given $i^{(m)}$, the effective periodic rate is given by $j = \dfrac{i^{(m)}}{m}$.
- The effective annual rate i can be found by solving the equation $(1+i) = (1+j)^m$.

It is important to remember that nominal rates are simply a means of indirectly reporting the effective periodic rates. Nominal rates should almost never be used directly to calculate anything other than j.

Pay very careful attention to whether annual rates provided to you in a problem are nominal or effective. A very common mistake is to treat a nominal rate as an effective annual rate, or vice versa.

Example 1.35 Assume an account earns compound interest at an effective monthly rate of 1%.

a) Find the nominal annual rate, $i^{(12)}$.

b) Find the annual effective rate, i.

Example 1.36 A loan of 3500 collects interest at a nominal rate of $i^{(4)} = 5\%$. The loan is repaid with a single payment after 28 months. Find the size of the payment.

When asked to determine a nominal rate, you should start by finding the effective periodic rate and then scaling to a nominal annual rate by multiplying by *m*.

Example 1.37 A loan of 2400 collects interest at a nominal rate of $i^{(12)} = r\%$. The loan is repaid with a single payment of 3085.12 after 3 years. Find *r*.

Example 1.38 Joe and Mick simultaneously make investments into different funds. You are given the following information about the investments:

- Joe's fund earns interest at a nominal rate of 8% convertible quarterly.
- Mick's fund earns interest at a nominal rate of 6% compounded every six months.
- Joe invests 50 more than Mick.
- At the end of 5 years, the value of Joe's fund is twice the value of Mick's fund.

Determine the size of Mick's initial investment.

Although we will generally reserve the symbol *i* to refer to an annual effective rate, it does occasionally show up in problems in reference to nominal rates. It is important to read the problems carefully to make sure you understand what type of rates are being used.

Example 1.39 Steve and Tony each make investments at the same time. Steve invests *K* into a fund that accumulates interest at a nominal annual rate of *i*, convertible quarterly. Tony invests $2K$ into a fund that earns simple interest at an annual rate of *i*. Steve and Tony earn the same amount of interest during the last quarter of the sixth year. Find *i*.

Equivalent Rates

We have seen several ways of expressing interest rates for compound interest problems. We have worked with annual effective rates, continuously compounded rates, and nominal annual rates. These different types of rates are simply different ways of expressing the same type of growth. Every nominal rate has a unique annual effective rate that will produce the same growth. There will also be a unique continuously compounded rate that produces the same growth.

We say that two rates are **equivalent** if they have the same annual effective rate. Equivalent rates will produce the same accumulation factor over periods of equal length. This observation is useful for finding equivalent rates. For instance, a nomimal rate $i^{(2)}$ and a continuously compounded rate δ are equivalent if they produce the same accumulation over the course of one year, and thus satisfy the equation: $\left(1 + i^{(2)} / 2\right)^2 = e^{\delta}$

As an example, consider the nominal annual rate $i^{(2)} = 6\%$. This nominal rate corresponds to an effective semi-annual rate of $j = 3\%$. The equivalent annual effective rate can be found by solving $(1.03)^2 = 1+i$, which yields $i = 6.09\%$. We can calculate the force of interest to be $\delta = 5.9118\%$ by solving the equation $(1.03)^2 = e^{\delta t}$. The rates $i^{(2)} = 6\%$, $j = 3\%$, $i = 6.09\%$, and $\delta = 5.9118\%$ all describe the same rate of growth, and are thus considered to be equivalent.

Example 1.40

Fund A collects interest at a nominal annual rate of 10% convertible quarterly. Fund B collects interest at a nominal annual rate of *r*% convertible semiannually. The rates for the two funds are equivalent. Find *r*.

It can seem overwhelming to work with so many different types of rates when they are just different ways of expressing the same thing. However, each type of rate we have considered has practical uses.

Compounding Periods Less Frequent than One Year

It is possible (though rare) to encounter compound interest problems in which interest is compounded less frequently than once a year. For example, a problem might state that interest is compounded once every two years, every three years, etc. In such problems, the rate will generally be expressed as an nominal annual rate in which the rate has been scaled down to a year. Although this notation is nonstandard, it would be consistent to denote a nominal rate for an account that compounds interest every M years as $i^{(1/M)}$.

For instance, assume that an account accumulates interest at a nominal annual rate of 5%, compounded every three years. Then $i^{(1/3)} = 5\%$. The effective three-year rate would then be $j = 15\%$. The annual effective rate could be found by solving the equation $(1 + i)^3 = (1.15)$. Doing so yields $i = 4.769\%$.

Example 1.41

An account earns interest at a nominal annual rate of $i^{(1/2)} = 10$ % compounded every 2 years.

a) Find the effective 2 year rate.
b) Find the value of an investment of 100 after six years.
c) Find the value of an investment of 100 after three years.
d) Find the effective annual rate.
e) Find an equivalent nominal semiannual rate.

Example 1.42

The interest rate on a four year investment varies from year to year as follows:

i) During year 1, the fund earns an effective annual rate of 7%.

ii) During year 2, the fund earns a continuously compounded rate of 8%.

iii) During year 3, the fund earns a nominal annual rate of 9% compounded semiannually.

iv) During year 4, the fund earns a nominal annual rate of 5% convertible every two years.

Determine the effective annual rate of interest over the four year period.

1.6 RATES OF DISCOUNT

Rates of Discount

We have already seen several different methods of representing a rate of compound interest. In this section, we will learn about one more type of rate, called a **discount rate**.

When calculating the percentage difference between two values, we will obtain different results depending on which of the two values we use as the "base" value. For example, we can say that 150 is 50% more than 100, but it would also be correct to say that 100 is (roughly) 33% less than 150. Put another way:

- Changing from 100 to 150 represents a 50% increase.
- Changing from 150 to 100 represents a 33% decrease.

Whereas interest rates are rates of increase, discount rates represent rates of decrease.

Rates of Discount

- Assume an account grows according the the accumulation function $a(t)$.
- Recall that the annual effective rate of interest during year *t* is given by $i_t = \dfrac{a(t) - a(t-1)}{a(t-1)}$.
- The annual effective rate of discount during year *t* is defined as $d_t = \dfrac{a(t) - a(t-1)}{a(t)}$.

Example 1.43 Assume a fund is worth 80 at the beginning of year three and is worth 100 at the end of year three. Find the effective annual rates of interest and discount during year three.

Discount Rates For Compound Interest

We have seen in the past that the annual effective rate of interest i_t can vary over time when working with a general accumulation function $a(t)$. The same is true for discount rates d_t. However, when working with compound interest, both of these rates are constant. As usual, let *i* represent this constant annual effective rate of interest. We will now use *d* to denote the constant **annual effective rate of discount**. There are many useful relationships between *i*, *d*, and *v*.

Using the formulas above, we can derive a relationship for between the rates *i* and *d*. The derivation is as follows:

$$d = \frac{(1+i)^t - (1+i)^{t-1}}{(1+i)^t} = 1 - \frac{1}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} = \frac{1+i-1}{1+i} = \frac{i}{1+i}$$

One of the most important aspects of the discount rate is its relationship with the present value factor, *v*. A present value factor of $v = 0.97$ corresponds to an interest rate of $i = 0.0309278$ (verify this on your own). We can use the formula above to determine that the associated discount rate is $d = 0.03$. This hits that *d* and *v* are related by the expression $v = 1 - d$. We can verify this fact as follows:

$$1 - d = 1 - \frac{i}{1+i} = \frac{1+i}{1+i} - \frac{i}{1+i} = \frac{1+i-i}{1+i} = \frac{1}{1+i} = v$$

We will now summarize what we know about discount rates for compound interest. We will also state a few identities that we have not yet discussed, but can be useful at times.

Identities Involving Discount Rates

The following identities hold true for *i*, *d*, and *v*.

- $i = \dfrac{d}{1-d}$
- $d = \dfrac{i}{1+i}$
- $v = 1 - d$
- $d = i\,v$
- $i - d = i\,d$
- $\dfrac{1}{d} - \dfrac{1}{i} = 1$

Example 1.44 Find the annual effective interest rate i that is equivalent to an annual effective discount rate of $d = 8\%$. Also calculate the associated present value factor, v.

Accumulating With Discount Rates

The expression $v = 1 - d$ tells us how to convert easily between discount rates and present value factors, but it also gives us a means of accumulating using discount rates. If $v = 1 - d$, then $v^t = (1-d)^t$. It then follows that $a(t) = (1+i)^t = 1 / v^t = v^{-t} = (1-d)^{-t}$. We state this result along with some related results below.

Accumulating and Discounting

Assume that an account earns compound interest at an annual effective rate of i. Let d be the equivalent rate of discount.

- The accumulation function for the account can be written in either of the following forms
  $a(t) = (1+i)^t$ or $a(t) = (1-d)^{-t}$
- To accumulate a payment forward one year, we can **multiply** it by $1+i$ or **divide** it by $1-d$.
- To discount a payment back one year, we can **divide** it by $1+i$ or **multiply** by $1-d$.

Example 1.45 Find the present value of the following sequence of payments: A payment of 300 at the end of year 1, a payment of 500 at the end of year 3, and a payment of 200 at the end of year 5. Assume an annual effective discount rate of $d = 6\%$.

Example 1.46 An account earns a 4% annual effective rate of interest during year 1 and a 7% annual effective rate of discount during year 2. Find the effective annual rate of interest during the 2 year period.

Nominal Discount Rates

As with interest rates, discount rates can be expressed in terms of a non-annual period. Such non-annual periodic discount rates are stated in the form of scaled **nominal annual discount rates.**

Nominal Annual Rates of Discount

- The **nominal annual discount rate compounded (or convertible) *m*-thly** is denoted by $d^{(m)}$.
- Given $d^{(m)}$, the effective periodic rate is given by $k = \frac{d^{(m)}}{m}$.
- The effective annual rate i can be found by solving the equation $(1+i) = (1-k)^{-m}$.

Example 1.47

Helga invests 5000 into an account. The account earns a nominal annual discount rate of 8% convertible quarterly during the first three years, and a nominal annual discount rate of 6% convertible quarterly during all later years.

a) Determine the value of the account at the end of 8 years.
b) Determine the annual effective rate of interest earned by the account during the first 8 years.

Example 1.48

An investment of 1000 is made to a fund. The fund earns a nominal annual discount rate of r convertible quarterly during the first year, and a nominal annual interest rate of r during the second year. The value of the account at the end of two years is 1185.34. Find r.

Simple Discount Rates

It is also possible to define the concept of simple discount rates. With simple interest, we assume that the accumulation factor scales linearly with time. With simple discount, we assume that it is, instead, the discount factor that scales linearly with time. For example, a simple discount rate of 3% would result in a one year discount factor of 0.97, a two year discount factor of 0.94, a three year discount factor of 0.91, and so on. The discount factor will decrease by 0.03 for each additional year that you wish to discount.

Rates of Simple Discount

- Let d be an annual rate of simple discount.
- The simple discount accumulation function defined by d is given by $a(t) = (1 - dt)^{-1}$.

Example 1.49

An amount of P is loaned at a simple discount rate of 7%. The loan is repaid after 3 years with a single payment of 1000. Find P.

It should be noted that while annual effective interest rates and annual effective discount rates for compound interest are two ways of expressing the exact same idea, the concepts of simple interest and simple discount are quite different from one another.

1.7 ZERO-COUPON BONDS AND T-BILLS

Bonds

It is often necessary for corporations and governments (both local and federal) to borrow money in order to fund projects. A corporation might need to borrow money in order to develop a new product, to build a new office building, to acquire another company, or any number of other reasons. Governments often need to borrow money to pay for various social programs, or to fund new infrastructure projects, such as building roads, bridges, airports, and stadiums. When a corporation or government needs to borrow large amounts of money, they generally borrow that money from the public in the form of **bonds**. The entity will split the amount they wish to borrow into many individual bonds, which investors can purchase, thereby making a small loan to the entity. When the bonds are issued, they include terms that describe how the debt will be repaid to the investor.

Assume that a corporation needs to raise $410,000 to provide initial funding for a project they are pursuing. They decide to do this by issuing 500 bonds. Each bond has a price of $820 and is repaid by the corporation with a single payment of $1000 at the end of 10 years. If the corporation sells all 500 bonds, then they will have raised the $410,000 they desired, and will settle their debt by paying a total of $500,000 in 10 years. In this scenario, the corporation will be paying an annual effective rate of interest that is slightly higher than 2%. The purchasers of the bonds will earn that same rate on their investments.

The scenario detailed above is an example of a **zero-coupon bond**. This means that the bond is settled with a single payment made by the borrower at some point in the future. Some bonds will make regular interest payments, called **coupons**, at regular intervals throughout the life of the loan. We will restrict our current discussion to zero-coupon bonds, but will discuss coupon-paying bonds in detail in a later section. When a bond is repaid by the borrower, it is said to have **matured**. The amount repaid to the investor is called the **maturity amount** of the bond.

Some short-term bonds (called T-bills) are priced using simple interest or simple discount formulas. We will study T-bills later in this section. Unless you are told specifically that you are working with a T-bill, you should assume compound interest is being used to price a bond. When working with such bonds, it is common to quote their rates using nominal annual rates compounded semiannually.

Example 1.50 A company issues a new zero-coupon bond. The bond is set to mature for 1000 in 15 years. The bond is priced to yield its purchaser a nominal annual rate of 3% convertible semiannually. Determine the price of the bond.

Example 1.51 A city government issues a 30-year zero-coupon bond. The bond matures for 1000 and has a price of 400. Determine the yield rate on the bond. Express your answer as a nominal annual rate of interest compounded semiannually.

Treasury Bills

The term **treasury bill**, or **T-bill**, refers to a short-term zero-coupon bond issued by the U.S. and Canadian treasuries. T-bills are available with terms of 4 weeks, 13 weeks, 26 weeks, or 52 weeks. T-bills differ from other types of zero-coupon bonds in that they are not priced using compound interest formulas. U.S. T-bills are priced using a variation of the simple discount formula, whereas Canadian T-bills are priced using the simple interest formula.

United States Treasury Bills

Quoted rates for U.S. Treasury bills are determined using the simple discount accumulation function, which is given by $a(t) = (1 - dt)^{-1}$. If P is the price of the bond, F is the maturity amount, and d is the quoted discount rate, then $F = P \cdot (1 - dt)^{-1}$ and $P = F \cdot (1 - dt)$.

There is one unusual aspect of U.S. T-bills that we need to be aware of. The pricing formula for U.S. Treasury bills assumes a 360-day year instead of a standard 365-day year. When working with (for example) a 4-week T-bill, one would use $t = 28 / 360$ in the pricing formula. For a 26-week T-bill, one would use $t = 182 / 360$.

United States Treasury Bills

- Assume a U.S. T-bill has a price of *P*, a maturity amount of *F*, and a quoted rate of *d*.
- Let *t* be the term of the bond in years, assuming a 360-day year.
- Then we have $P = F \cdot (1 - dt)$.
- We can also show that $d = \dfrac{360}{\text{Days to Maturity}} \cdot \dfrac{I}{F}$, where $I = F - P$.

Example 1.52 A new U.S. Treasury bill has a price of 990 and matures in 26 weeks for 1000. Find the quoted rate for the T-bill, as well as the annual effective rate of interest paid by the bond.

Example 1.53 A new U.S. Treasury bill matures in 13 weeks for 1000 and has a quoted rate of 2%. Find the price of the T-bill.

Canadian T-Bills

Canadian T-bills are priced using the simple interest accumulation function, and use a 365-day year. Formulas related to Canadian T-bills are provided below.

Canadian Treasury Bills

- Assume a Canadian T-bill has a price of *P*, a maturity amount of *F*, and a quoted rate of *i*.
- Let *t* be the term of the bond in years, assuming a 365-day year.
- Then we have $F = P \cdot (1 + it)$.
- We can also show that $i = \dfrac{365}{\text{Days to Maturity}} \cdot \dfrac{I}{P}$, where $I = F - P$.

Example 1.54 A new Canadian Treasury bill has a price of 982 and matures in 52 weeks for 1000. Find the quoted rate for the T-bill, as well as the annual effective rate of interest paid by the bond.

Example 1.55 A new Canadian Treasury bill matures in 4 weeks for 1000 and has a quoted rate of 3%. Find the price of the T-bill.

1.8 FORCE OF INTEREST

Definition of Force of Interest

When working with compound interest, whether compounded continuously, annually, or on some non-annual basis, the effective annual interest rate is the same over any two periods of the same length. For general accumulation functions however, we have seen that the annual effective interest rate can (and likely will) vary when calculated over different periods of the same length. This fact is demonstrated in the next example.

Example 1.56 A fund grows according to the accumulation function $a(t) = 1+0.06t^2$. Find the annual effective interest rate during each quarter of the first year.

In the previous example, the rate at which interest is being accumulated isn't just changing from one quarter to the next, but is in fact changing from one instant to the next. In this section, we will introduce a mechanism for discussing continuously changing rates of growth. This tool will be called the force of interest.

Assume that we have an account whose value at time t is given by the amount function $A(t)$. We know from calculus that the derivative $A'(t)$ provides the instantaneous rate of growth in the value of the fund. The quantity $A'(t)$ provides a *absolute growth rate*, which is measured in units of currency per year. This is not a great tool for measuring a fund's performance, as it doesn't take into account the current value of the fund. For example, assume we know that an account has an instantaneous absolute growth rate of $A'(t) = 50$ units of currency per year at time *t*. That relative rate of growth is much more substantial if the current value of the account is $A(t) = 100$ than if the current value is $A(t) = 2000$.

Rather than an absolute growth rate, we would prefer to work with a *relative growth rate* that expresses the absolute growth rate as a proportion of the current value of the account. The expression $A'(t)/A(t)$ provides us with such a relative growth rate. We will refer to $A'(t)/A(t)$ as the **force of interest**, and will denote it by δ_t.

Let $a(t)$ be the accumulation function associated with the amount function $A(t)$. Then $A(t) = P \cdot a(t)$, where P is the initial principal in the fund. Noting that $A'(t) = P \cdot a'(t)$, we can show that $\delta_t = a'(t)/a(t)$ as follows:

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{P \cdot a'(t)}{P \cdot a(t)} = \frac{a'(t)}{a(t)}$$

Definition of Force of Interest

- Let $A(t)$ and $a(t)$ be the amount and accumulation functions for a fund, respectively.
- The **force of interest** of the fund at time *t* is defined to be $\delta_t = \dfrac{A'(t)}{A(t)} = \dfrac{a'(t)}{a(t)}$.

Note that, in general, the force of interest is a function of *t* and will change continuously over time.

Example 1.57 Find the force of interest δ_t for the accumulation function $a(t) = 1 + 0.06t^2$. Calculate δ_0, $\delta_{0.25}$, $\delta_{0.5}$, $\delta_{0.75}$, and δ_1.

Force of Interest as a Nominal Rate

The force of interest provides a means of measuring a continuously changing rate of growth associated with an accumulation function $a(t)$. The actual value of δ_t can be interpreted as a nominal annual rate of interest convertible continuously. To see that this is true, we will need to work with limits.

Let m be some positive number, and let $\Delta t = 1/m$. That is, Δt is a length of time equal to one *m*th of a year. Let $j_t^{(m)}$ equal the (non-annual) effective rate of interest during the period of time $[t,\ t+\Delta t]$. Then $j_t^{(m)}$ is equal to the amount of growth during this period divided by the value at the beginning of the period. That is:

$$j_t^{(m)} = \frac{a(t+\Delta t) - a(t)}{a(t)}$$

Now, let $i_t^{(m)}$ be the nominal annual rate convertible *m*-thly, calculated over the period $[t,\ t+\Delta t]$. Then we have:

$$i_t^{(m)} \;=\; j_t^{(m)} \cdot m \;=\; j_t^{(m)} \cdot \frac{1}{\Delta t} \;=\; \frac{a(t+\Delta t) - a(t)}{a(t)} \cdot \frac{1}{\Delta t} \;=\; \frac{a(t+\Delta t) - a(t)}{\Delta t} \cdot \frac{1}{a(t)}$$

Now define $i_t^{(\infty)}$ to be $i_t^{(\infty)} = \lim_{m\to\infty} i_t^{(m)}$. Note that as m approaches infinity, Δt approaches zero. We now have:

$$i_t^{(\infty)} \;=\; \lim_{m\to\infty}\left[\frac{a(t+\Delta t) - a(t)}{\Delta t} \cdot \frac{1}{a(t)}\right] \;=\; \lim_{\Delta t\to 0}\left[\frac{a(t+\Delta t) - a(t)}{\Delta t}\right] \cdot \frac{1}{a(t)} \;=\; a'(t) \cdot \frac{1}{a(t)} \;=\; \frac{a'(t)}{a(t)} \;=\; \delta_t$$

This verifies our earlier claim that δ_t is essentially a nominal annual rate of interest convertible continuously.

We will now consider additional examples related to the force of interest.

Example 1.58

Find the force of interest δ_t for each of the following accumulation functions.

a) $a(t) = 1 + 0.08t$

b) $a(t) = (1 + 0.01t)^2$

c) $a(t) = (1.05)^t$

d) $a(t) = e^{0.08t}$

Example 1.59

The value of an account at time *t* is given by the amount function $A(t) = 2000(1 + 0.05t)^2$.

a) Calculate the force of interest at the end of the first year.

b) Find the time at which the force of interest is equal to 0.08.

Example 1.60

The value of an account at time t is given by $A(t) = Pt^2 + Qt + R$. You are given that $A(0) = 200$, $A(1) = 232$, and $A(3) = 368$. Calculate the force of interest at time $t = 0.5$.

Finding the Accumulation Function for a Given Force of Interest

We have seen how to find the force of interest for a given accumulation function $a(t)$. However, most problems that you will encounter involving a force of interest will provide you with δ_t, from which you will need to reconstruct an accumulation function $a(t)$. To see how to do this, first observe that $\delta_t = a'(t)/a(t) = \frac{d}{dt}\ln[a(t)]$. Switching to a dummy variable *r* and then integrating from 0 to *t* gives us $\int_0^t \delta_r\,dr = \int_0^t \frac{d}{dr}\ln[a(r)]\,dr$. We can apply the fundamental theorem of calculus to the right hand side to obtain $\int_0^t \delta_r\,dr = \ln[a(t)]$. Exponentiating both sides then gives us the expression $a(t) = e^{\int_0^t \delta_r\,dr}$.

Finding an Accumulation Function Given a Force of Interest

- Given a force of interest δ_t, we can find the associated accumulation function using the formula $a(t) = e^{\int_0^t \delta_r \, dr}$.

Example 1.61

An account earns interest according to the force interest $\delta_t = \dfrac{0.2}{1 + 0.1t}$, for $t \geq 0$.

a) Find $a(t)$.
b) Assume 50 is deposited at time 0. Find the accumulated value at time 6.
c) Find the present value of a payment of 200 at time 5.
d) Find the present value of a payment of 200 at time 5 and a payment of 100 at time 10.
e) An amount of P is deposited at time 0. The account is worth 500 at time 5. How much is the account worth at time 8?

Example 1.62

An amount of P is deposited into each of two funds: Fund X and Fund Y.

- Fund X accumulates at a force of interest given by $\delta_t = \dfrac{t^2}{K}$.
- Fund Y earns a nominal annual rate of interest of 10% convertible semiannually

At the end of year 6, the accumulated values of the two funds are equal. Find K.

Finding Accumulation Functions Algebraically

Assume that you are provided with a force of interest that is written as a fraction where the numerator is equal to the derivative of the denominator. That is, assume that $\delta_t = f'(t) / f(t)$ for some function $f(t)$. Since δ_t is defined as $\delta_t = a'(t) / a(t)$, it would be tempting to assume that $f(t) = a(t)$ in this situation. That is not necessarily true, however. Assume that $a(t) = K \cdot f(t)$ for some constant K. Then $a'(t) = K \cdot f'(t)$, and thus $\delta_t = a'(t) / a(t) = f'(t) / f(t)$ even though $a(t) \neq f(t)$. It can, however, be shown that if $\delta_t = f'(t) / f(t)$, then $a(t) = K \cdot f(t)$ for some constant K. The actual value of K can be determined using the property that $a(0) = 1$.

Example 1.63

Find the accumulation functions associated with the following forces of interest.

a) $\delta_t = 0.04 / (1 + 0.04t)$
b) $\delta_t = 1 / (t + 8)$
c) $\delta_t = 0.1t / (1 + 0.05t^2)$
d) $\delta_t = t / (4 + 0.5t^2)$

Example 1.64

Phillip and Gary each deposit an amount P into separate funds. Phillip's fund earns a nominal annual rate of discount of 8% convertible quarterly. Gary's fund accumulates interest at a force of interest $\delta_t = 1 / (t + 10)$. After 6 years, Phillip's fund is worth 2500 and Gary's fund is worth K. Find K.

Example 1.65 An account earns interest according to the force of interest $\delta_t = 2k / (1+kt^2)$. The annual effective rate of interest earned by the account during year 6 is equal to 10%. Determine the force of interest at end end of year 6.

This algebraic method can, in theory, be used to find the accumulation function associated with any force of interest. However, as a result of simplification, the force of interest is often written in a way that is not immediately recognizable as being in the form $\delta_t = f'(t) / f(t)$. Consider the force of interest in Example 1.61. To use this algebraic method on this problem, once would have to recognize that $\delta_t = \dfrac{0.2}{1+0.1t} = \dfrac{0.2(1+0.1t)}{(1+0.1t)^2}$.

Force of Interest for Compound Interest

Consider the accumulation function $a(t) = (1+i)^t$, which is associated with compound interest with annual compounding. Calculating the force of interest for this accumulation function gives $\delta_t = \ln(1+i)$. Since i is a constant, the force of interest is constant for any compound interest accumulation function. For that reason, we will drop the subscript from δ_t when working with compound interest, simply stating that $\delta = \ln(1+i)$. It should be noted that the concept of force of interest for compound interest is exactly the same as the concept of a continuous rate of compound interest. If we are given a constant force of interest δ, our accumulation function could thus be written as $a(t) = e^{\delta t}$.

Example 1.66 Fund X accumulates interest at a force of interest of $\delta = r$. Fund Y earns simple interest at an annual rate of $i = r$. A deposit into Fund X will double in value over the course of 10 years. Determine how long it would take a deposit into Fund Y to double in value.

Example 1.67 The interest rate earned by an account varies each year, as follows:

- A nominal annual discount rate of 8% convertible quarterly is earned during year 1.
- A nominal annual interest rate of 6% convertible every 2 years is earned during year 2.
- A force of interest of 5% is earned during year 3.

Find the annual effective rate of interest during the three year period.

Accumulating over General Periods

As discussed in Section 1.3, if we are not working with compound interest, we would use the accumulation factor $a(t_1, t_2) = a(t_2) / a(t_1)$ to accumulate over a period from time t_1 to time t_2.

Example 1.68 An account accumulates at a force of interest $\delta_t = \dfrac{1}{1+t}$, $t \geq 0$. Assume 100 is deposited at time 3. Find the value of the account at time 5.

Example 1.69 A fund earns interest at a force of interest given by $\delta_t = \dfrac{0.06t}{1+0.03t^2}$. A deposit of 200 is made into the fund at time 0, and another deposit of 100 is made at at the end of year 5.

a) Find the value of the fund at the end of year 8.

b) Find the time t at which the value of the fund is equal to 1000.

CHAPTER 2 – Annuities

2.1 ANNUITY IMMEDIATE

An *annuity* is any series of periodically occurring payments. Annuities are an important concept and are frequently encountered in the fields of finance and economics. Before we begin our discussion of the mathematics related to annuities, we need to review the concept of a finite geometric series.

Finite Geometric Series

A **finite geometric sequence** is an ordered sequence of n numbers in which the ratio of any two consecutive numbers in the sequence is constant. If a is the first term in the sequence, and r is the common ratio, then the terms of the sequence would be $a,\ ar,\ ar^2,\ ar^3,\ \ldots,\ ar^{n-1}$.

A **finite geometric series** is the sum of a finite geometric sequence. A geometric series with initial term a, common ratio r, and with n total terms would have the form $a + ar + ar^2 + \ldots + ar^{n-1}$. A formula for the sum of a finite geometric series is given below.

Sum of a Finite Geometric Series

- Consider a **finite geometric series** of the form $a + ar + ar^2 + \ldots + ar^{n-1}$.
- The sum of this series is given by $a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a - ar^n}{1-r} = \dfrac{a\left(1-r^n\right)}{1-r}$.
- A convenient way of remembering the formula for the sum of a finite geometric series is to read the $\dfrac{a - ar^n}{1-r}$ version of the formula as $\dfrac{(\text{first term}) - (\text{first ommitted term})}{1 - (\text{common ratio})}$.

Example 2.1

Find the sum of the following finite geometric series.

a) $3 + 6 + 12 + 24 + 48$

b) $1 + (1.06) + (1.06)^2 + (1.06)^3 + \ldots + (1.06)^{10}$

Level Annuities

The payments in an annuity might be fixed at some level, might increase or decrease in some prescribed way, or might follow some other type of pattern. We will begin our study of annuities by considering those whose payments remain constant in size. Such annuities are called **level annuities**.

The payments in any annuity can be considered to be occurring either at the beginning of a time period, or at the end of a time period. If we are treating the payments as occurring at the end of a time period, then we call the annuity an **annuity immediate**. If, on the other hand, we consider the payments to be taking place at the beginning of a time period, we call the annuity an **annuity due**. We will first look at annuities immediate.

Annuity Immediate

Consider an annuity at makes payments of the end of each year for a total of n years. Since the payments occur at the end of the year, this annuity would be considered an **annuity immediate**. Note that the first payment would occur at time $t = 1$.

We begin by assuming that each of the payments is equal to 1. We wish to develop formulas for the present value of such an annuity at time $t = 0$, as well as its accumulated value at time $t = n$. Actuaries use special notation for the present value and accumulated value of annuity of this form.

- $a_{\overline{n}|i}$ denotes the PV at time $t = 0$ of an annuity that pays 1 at the end of each year for n years.
- $s_{\overline{n}|i}$ denotes the AV at time $t = n$ of an annuity that pays 1 at the end of each year for n years.

These symbols $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ are read as "*a* angle *n* i" and "*s* angle *n* i" respectively.

Using ideas from Chapter 1, we can show that $a_{\overline{n}|i} = v + v^2 + ... + v^n$ and $s_{\overline{n}|i} = 1 + (1+i) + (1+i)^2 ... + (1+i)^{n-1}$. Both of these series are finite geometric series, the first with common ratio v and the second with common ratio $(1 + i)$. We can apply the formula for the sum of a finite geometric series, along with relationships between i and v to obtain the two formulas below, which are fundamental to the study of annuities:

$$a_{\overline{n}|i} = \frac{1 - v^n}{i} \quad \text{and} \quad s_{\overline{n}|i} = \frac{(1 + i)^n - 1}{i}$$

The annuity as a whole has the same time value as a single payment of $a_{\overline{n}|i}$ at time $t = 0$, and also the same time value as a single payment of $s_{\overline{n}|i}$ at time $t = n$. Such payments of $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ would thus have equal time values. It follows that the two values are related by the expressions $s_{\overline{n}|i} = (1 + i)^n a_{\overline{n}|i}$, and $a_{\overline{n}|i} = v^n s_{\overline{n}|i}$.

If an annuity immediate has level annual payments of R instead of 1, then its present and accumulated values are greater by a factor of R, and are thus given by $R a_{\overline{n}|i}$ and $R s_{\overline{n}|i}$ respectively.

Annuities Immediate

- Consider an annuity immediate that makes payments of 1 at the end of each year.
- The present value of such an annuity at time $t = 0$ is denoted by $a_{\overline{n}|i}$ and calculated using:
  $a_{\overline{n}|i} = v + v^2 + ... + v^n = \dfrac{1 - v^n}{i}$
- The accumulated value of the annuity at time $t=n$ is denoted by $s_{\overline{n}|i}$ and calculated using:
  $s_{\overline{n}|i} = 1 + (1+i) + (1+i)^2 ... + (1+i)^{n-1} = \dfrac{(1 + i)^n - 1}{i}$
- The values $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ are related by $s_{\overline{n}|i} = (1 + i)^n a_{\overline{n}|i}$, and $a_{\overline{n}|i} = v^n s_{\overline{n}|i}$.
- The present and accumulated values of an annuity immediate that pays R at the end of each year are given by $R a_{\overline{n}|i}$ and $R s_{\overline{n}|i}$ respectively.

It is important to remember the following facts about the times associated with the values $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$:

- The value $a_{\overline{n}|i}$ gives the time value of the annuity **one year before** the first payment.
- The value $s_{\overline{n}|i}$ gives the time value of the annuity **at the time of** the last payment.

When only a single interest rate is being considered, we will often use the shorthand notations $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

Example 2.2 Deposits of 75 are made into a fund at the end of each year for 24 years. The effective annual interest rate is 3%. Calculate the present value of this series of payments.

Example 2.3 Deposits of 120 are made into a fund at the end of each year for 8 years. The effective annual interest rate is 4.5%. Calculate the accumulated value of the series of payments at the end of the 8th year.

Example 2.4 Deposits of P are made into a fund at the end of each year for 12 years. At an effective annual interest rate of 6%, the accumulated value of the series of payments at the end of the 12th year is 2783.54. Find P.

Performing Annuity Calculations with the BA II Plus

Financial calculators such as the BA II Plus come equipped with "Time Value of Money (TVM) Calculators" for performing annuity calculations. Some brief examples of how to use the BA II Plus to perform annuity calculations are given below. I suggest reading the manual or watching YouTube videos to get full instructions.

1. To calculate $100\,a_{\overline{16}|5\%}$ you would enter:
   [2ND] [CLR TVM] 16 [N] 5 [I/Y] 100 [+/-] [PMT] [CPT] [PV]

2. To calculate $100\,s_{\overline{16}|5\%}$ you would enter:
   [2ND] [CLR TVM] 16 [N] 5 [I/Y] 100 [+/-] [PMT] [CPT] [FV]

3. Assume you know that $800 = 100\,a_{\overline{16}|i}$ and you wish to find i. You would enter:
   [2ND] [CLR TVM] 16 [N] 800 [PV] 100 [+/-] [PMT] [CPT] [I/Y]

Although the BA II Plus can be used to calculate $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$, it is **strongly** recommended that you get used to performing such calculations using the TI-30X along with the formulas we have introduced. There are three reasons for this suggestion:

1. Once you get used to doing these calculations, you will almost certainly be able to perform them more quickly with the TI-30X than with the BA II Plus, and will likely make fewer errors.
2. It is necessary for you to learn the formulas $a_{\overline{n}|i} = \left(1 - v^n\right) / i$ and $s_{\overline{n}|i} = \left((1+i)^n - 1\right) / i$. The best way to learn these formulas is by using them.
3. Later in this chapter we will encounter more complicated types of annuities for which the BA II Plus can not easily be used to calculate their present or accumulated values. When working with such annuities, it will be helpful if you are comfortable with performing basic annuity calculations using the TI-30X.

That said, there are circumstances under which the BA II Plus is useful. Assume that you need to solve for the interest rate in an equation of the form $a_{\overline{n}|i} = K$. This would involve finding the roots of an nth degree polynomial. If n is greater than 3, then you will almost certainly require some form of technology to solve for i.

Solving for n in the equation $a_{\overline{n}|i} = K$ is certainly possible without the BA II Plus, but the process involves taking logarithms and it will likely be quicker to find n using the TVM calculator.

Example 2.5 Deposits of 60 are made into a fund at the end of each year for 16 years. The present value of the series of payments is 596.08. Find the effective annual interest rate.

Example 2.6

Amber and Bert each make deposits of 500 at the end of each year for 20 years. Amber's account earns an annual effective rate of 4% and Bert's account earns an annual effective rate of 6%.

After making deposits for 20 years, both people begin withdrawing money from their accounts. They each make withdrawals at the end of the year for 12 years, with the first withdrawal occurring exactly one year after the last deposit. Amber's withdrawals are in an amount of P each, while Bert's are each in an amount of Q.

a) Determine the value of each person's account at the end of 20 years.
b) Find P and Q.

Annuities with Non-Annual Payments

In our discussion of annuities, we have so far assumed that the payments were made annually. We will often encounter annuities where the payments are made every month, every 6 months, or quarterly. We can use the same formulas to calculate the PV and AV of these annuities. When dealing with non-annual payment periods, the n in the formulas for $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ should equal the total number of payment periods and i should be replaced with the effective periodic rate, j.

Example 2.7

Gary deposits P at the end of each month. His employer matches each deposit. His fund earns interest at a rate of 3% convertible monthly.

Gary makes deposits for 30 years, and then retires. After retirement, he withdraws \$2000 at the end of each month for 20 years. After these 20 years, the account is empty.

Find P.

It is also possible to encounter annuities in which the payments occur less frequently than once a year, as seen in the next example.

Example 2.8

An annuity pays 50 every two years, with the first payment occurring at the end of year 1 and the last payment occurring at the end of year 29. Assume an annual effective interest rate of 5%.

a) Find the present value of this annuity.

b) Find the accumulated value of this annuity at the time of the last payment.

2.2 ANNUITY DUE

Annuity Due

As mentioned in the last section, we will occasionally wish to consider annuities in which the payments occur at the beginning of the year rather than at the end of the year. Such annuities are called **annuities due**. We introduce the following notation to refer to the present and accumulated values of an annuity due:

- $\ddot{a}_{\overline{n|}i}$ denotes the PV at time $t = 0$ of an annuity that pays 1 at the beginning of each year for n years.
- $\ddot{s}_{\overline{n|}i}$ denotes the AV at time $t = n$ of an annuity that pays 1 at the beginning of each year for n years.

We can use principles from Chapter 1 to show that $\ddot{a}_{\overline{n|}i}$ and $\ddot{s}_{\overline{n|}i}$ can be written as $\ddot{a}_{\overline{n|}i} = 1 + v + v^2 + ... + v^{n-1}$ and $\ddot{s}_{\overline{n|}i} = (1+i) + (1+i)^2 ... + (1+i)^n$. Applying formulas for the sum of a finite geometric sequence, as well as identities relating *i*, *v*, and *d*, we obtain the following formulas for $\ddot{a}_{\overline{n|}i}$ and $\ddot{s}_{\overline{n|}i}$:

$$\ddot{a}_{\overline{n|}i} = \frac{1 - v^n}{d} \quad \text{and} \quad \ddot{s}_{\overline{n|}i} = \frac{(1+i)^n - 1}{d}$$

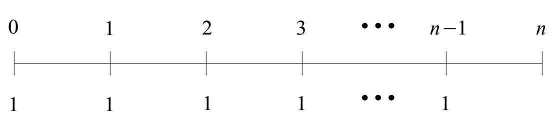
If the annual payments are in an amount of R rather than 1, then the present and accumulated values of the annuity due are given by $R\ddot{a}_{\overline{n|}i}$ and $R\ddot{s}_{\overline{n|}i}$ respectively.

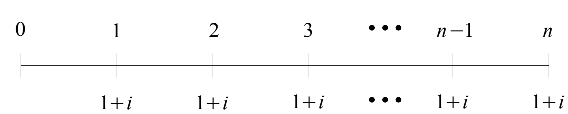
Notice that the formulas above are nearly identical to the formulas for $a_{\overline{n|}i}$ and $s_{\overline{n|}i}$. The only difference is that the denominator is equal to i in the formulas for annuities immediate, and equal to d for annuities due. This presents a convenient mnemonic device: "Use *i* for annuities **(i)**mmediate, and use *d* for annuities **(d)**ue.**"**

While you should certainly memorize the formulas presented above for $\ddot{a}_{\overline{n|}i}$ and $\ddot{s}_{\overline{n|}i}$, they are not generally the most efficient way to perform calculations relating to annuities due. These formulas require you to calculate the discount rate *d*. Although that is is not a difficult task, it does introduce a second rate that you have to keep track of. An alternate method of calculating $\ddot{a}_{\overline{n|}i}$ and $\ddot{s}_{\overline{n|}i}$ involves appealing to relationships between annuities due and immediate. We now discuss one such relationship.

Annuity Immediate as Delayed Annuity Due

Notice that a payment of 1 at the beginning of a year has the same time value as a payment of $(1+i)$ at the end of the year. Given an annuity due paying 1 at the beginning of each year, we could postpone each payment, replacing them with payments of $(1+i)$ at the end of the year. In this way, we can convert an annuity due with payments of 1 into an annuity immediate with payments of $(1+i)$. These two annuities are illustrated in the time diagrams below. These annuities are equivalent, and will have the same time value at all times. This allows us to conclude that $\ddot{a}_{\overline{n|}i} = (1+i)a_{\overline{n|}i}$ and $\ddot{s}_{\overline{n|}i} = (1+i)s_{\overline{n|}i}$.





Using the formulas $\ddot{a}_{\overline{n|}i} = (1+i)a_{\overline{n|}i}$ and $\ddot{s}_{\overline{n|}i} = (1+i)s_{\overline{n|}i}$ to find $\ddot{a}_{\overline{n|}i}$ and $\ddot{s}_{\overline{n|}i}$ saves us from needing to calculate the discount rate, *d*. These formulas also provide us with a method of using the BA-II Plus to calculate values such as $R\ddot{a}_{\overline{n|}i}$. Notice that $R\ddot{a}_{\overline{n|}i} = R(1+i)a_{\overline{n|}i}$. We can use the BA-II Plus to calculate the right-hand side of this equation by entering $R(1+i)$ as the payment. As similar approach can be used to calculate $R\ddot{s}_{\overline{n|}i}$.

We now summarize what we have learned about annuities due.

Annuities Due

- Consider an annuity due that makes payments of 1 at the beginning of each year for n years.
- The present value of such an annuity at time $t = 0$ is denoted by $\ddot{a}_{\overline{n}|i}$ and is equal to:
  $$\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + ... + v^{n-1} = \frac{1 - v^n}{d}$$
- The accumulated value of the annuity at time $t=n$ is denoted by $\ddot{s}_{\overline{n}|i}$ and is equal to:
  $$\ddot{s}_{\overline{n}|i} = (1+i) + (1+i)^2 ... + (1+i)^n = \frac{(1 + i)^n - 1}{d}$$
- The time values of annuities due and annuities immediate are related by the following equations: $\ddot{a}_{\overline{n}|i} = (1 + i)a_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i} = (1 + i)s_{\overline{n}|i}$.
- The present and accumulated values of an annuity due that pays R at the beginning of each year are given by $R\ddot{a}_{\overline{n}|i}$ and $R\ddot{s}_{\overline{n}|i}$, respectively.

Example 2.9 Deposits of 45 are made into a fund at the beginning of each year for 18 years. The effective annual interest rate is 7.5%. Calculate the present value of this series of payments.

Example 2.10 Deposits of 80 are made into a fund at the beginning of each year for 10 years. The effective annual interest rate is 5%. Calculate the accumulated value of the series of payments at the end of the 10th year.

Example 2.11 Deposits of P are made into a fund at the beginning of each year for 15 years. At an effective annual interest rate is 4.5%, the present value of the series of payments is 729.48. Find P.

The following problem can be easily solved using the BA-II Plus and the relationship $\ddot{a}_{\overline{n}|i} = (1 + i)a_{\overline{n}|i}$.

Example 2.12 Deposits of 110 are made into a fund at the beginning of each year for T years. At an effective annual interest rate is 8%, the present value of the series of payments is 979.42. Find T.

Time of First and Last Payments

The end of any one year can be thought of as the beginning of the following year. This observations allows us to think of any annuity immediate as an annuity due, and vice versa. In some sense, the distinguishing characteristic between an annuity immediate and an annuity due is simply a matter of whether we are selecting the time $t=0$ to be the time of the first payment, or one year prior.

Since the difference between these two types of annuities is primarily a matter of perspective, it would be useful to discuss the different results yielded by using one type of annuity over the other. These are summarized below.

- Annuities Immediate
  - $a_{\overline{n}|i}$ gives the present value of the annuity one period **BEFORE** the first payment.
  - $s_{\overline{n}|i}$ gives the accumulated value of the annuity at the **SAME TIME** as the last payment
- Annuities Due
  - $\ddot{a}_{\overline{n}|i}$ gives the present value of the annuity at the **SAME TIME** as the first payment.
  - $\ddot{s}_{\overline{n}|i}$ gives the accumulated value of the annuity one period **AFTER** the last payment.

It is important to remember these rules. Doing so will allow you a degree of flexibility in working with annuity problems. There are situations were it is more convenient to treat an annuity as an annuity due than as an annuity immediate, and vice versa.

Example 2.13 Deposits of 25 are made into a fund at the end of each year for 8 years with the first deposit occurring at t = 4. The effective annual interest rate is 6%. Calculate the present value of the series of payments.

Example 2.14 Deposits of 40 are made into a fund at the beginning of each year with the first deposit occurring at t = 8. The effective annual interest rate is 5%. Calculate the accumulated value of the series of payments at the end of the 26th year.

The "Plus One / Minus One" Formulas

We will now discuss a second relationship that exists between the formulas for annuities immediate and due. These new formulas are informally called the **Plus One / Minus One Formulas**. We will first state these formulas, and then discuss why they are true and when they should be used.

Plus One / Minus One Formulas

The following equations hold for all values of n and i

- $\ddot{a}_{\overline{n}|i} = a_{\overline{n-1}|i} + 1$
- $\ddot{s}_{\overline{n}|i} = s_{\overline{n+1}|i} - 1$

To see that $\ddot{a}_{\overline{n}|i} = a_{\overline{n-1}|i} + 1$ is true, note that $\ddot{a}_{\overline{n}|i}$ gives the present value of a sequence of n annual payments of 1, with the first payment occurring at time $t=0$. Imagine that we temporarily remove the payment at $t=0$. The remaining payments will occur at times $1, 2, 3, \ldots, n-1$, and can thus be thought of as an $(n-1)$-year annuity immediate. The present value of this annuity at time $t=0$ is given by $a_{\overline{n-1}|i}$. If we add back in the present value of the first payment, which is already at time $t=0$, then we get that the total present value for the n payments is equal to $a_{\overline{n-1}|i} + 1$. We thus conclude that $\ddot{a}_{\overline{n}|i} = a_{\overline{n-1}|i} + 1$.

A similar approach can be used to show that $\ddot{s}_{\overline{n}|i} = s_{\overline{n+1}|i} - 1$. Consider an annuity making n payments of 1 at times $0, 1, 2, \ldots, n-1$. The accumulated value of this annuity at time $t=n$, one year after the last payment, is $\ddot{s}_{\overline{n}|i}$. Now add another payment of 1 at time $t=n$. The total accumulated value of this sequence of n+1 payments at time $t=n$ is equal to $\ddot{s}_{\overline{n}|i} + 1$. But this new sequence forms an $(n+1)$-year annuity, which we are valuing at the time of the last payment. Thus the time $t=n$ accumulated value is also equal to $s_{\overline{n+1}|i}$. This tells us that $\ddot{s}_{\overline{n}|i} + 1 = s_{\overline{n+1}|i}$, and thus $\ddot{s}_{\overline{n}|i} = s_{\overline{n+1}|i} - 1$.

These formulas are particular useful if we need to solve for the interest rate in a problem involving an annuity due. To solve for a rate, we need to use the TVM calculator in the BA-II Plus. However, since the rate is unknown, the formulas $\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}$ and $\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}$ cannot help us in this scenario.

Example 2.15 An annuity makes annual payments of 2 at the beginning of each year for 10 years. The present value of the annuity is 16. Find the annual effective rate of interest.

Example 2.16 Assume an annual effective interest rate of i. At this rate, the present value of an n-year annuity immediate with annual payments of 500 is equal to 4559.29. At the same rate, the present value of an $(n-1)$-year annuity immediate with annual payments of 400 is equal to 3411.57. Find i.

2.3 PERPETUITIES

A perpetuity is an annuity that makes periodic payments forever. As with standard annuities, we will consider two types of perpetuities: perpetuities immediate that make payments at the end of each year, and perpetuities due that make payments at the beginning of each year.

Perpetuity Immediate

A perpetuity immediate is a perpetuity in which the first payment occurs one year after the creation of the perpetuity, with payments continuing annually forever. Equivalently, we can say that payments occur at the end of the year. The present value of a perpetuity immediate is denoted by $a_{\overline{\infty}|i}$, and is given by the infinite geometric series $a_{\overline{\infty}|i} = v + v^2 + v^3 + \ldots$. Using the formula for the sum of an infinite geometric series, we see that $a_{\overline{\infty}|i} = 1 / i$.

Example 2.17 An alumnus of State University wants to make a donation to fund an annual scholarship. The alumnus will deposit P into a fund earning a 4% annual effective rate of interest. The account will be used to fund annual scholarships of 2000, with the first scholarship to be awarded one year after the deposit. The scholarships are intended to last forever. Find P.

The payments in an annuity are equal to the interest earned by the account each year. Since the payments are exactly equal to the interest earned, the balance of the account is the same after each payment, and is equal to the initial principle. Thus, the payments will always be in the same amount, and since they do not decrease the amount of principal invested, they will last forever.

Example 2.18 At an annual effective interest rate of i, a 20-year annuity immediate with annual payments of 1429 has the same present value as that of a perpetuity immediate with annual payments of 1000. Find i.

Example 2.19 Jackie purchases a perpetuity immediate that pays 50 at the end of each year forever. Jackie pays P for the perpetuity, which would earn her an annual effective interest rate of 8% on her purchase.

Five years after purchasing the annuity, immediately after receiving the fifth payment, Jackie sells the perpetuity to Frankie for a price of Q. Taking into account her sale of the annuity, Jackie ultimately earned an annual effective rate of 7% on her original investment of P.

Determine the annual effective yield rate that Frankie would see on his investment of Q.

Perpetuity Due

A perpetuity due is a perpetuity in which the first payment occurs at the time of the creation of the perpetuity, with payments continuing annually forever. Equivalently, we can say that payments occur at the beginning of the year. The present value of a perpetuity due is denoted by $\ddot{a}_{\overline{\infty}|i}$, and is given by $\ddot{a}_{\overline{\infty}|i} = 1 + v + v^2 + v^3 + \ldots$. Summing this infinite geometric series yields the formula $\ddot{a}_{\overline{\infty}|i} = 1 / d$. As with annuities, it is generally easier to calculate the present value of a perpetuity due by exploiting relationships that exist between perpetuities immediate and due.

Relationships Between Perpetuity Immediate and Perpetuity Due

As with standard annuities, it is true that $\ddot{a}_{\overline{\infty}|i} = (1+i)a_{\overline{\infty}|i}$. This can be seen by noting that payments in a perpetuity immediate lag one year behind those those in a perpetuity due. This identity can also be established algebraically as follows: $\ddot{a}_{\overline{\infty}|i} = 1/d = (1+i)/i = (1+i)a_{\overline{\infty}|i}$.

Another useful relationship between the present value formulas for perpetuities immediate and due follows from the the facts that $a_{\overline{\infty}|i} = v + v^2 + v^3 + \ldots$ and $\ddot{a}_{\overline{\infty}|i} = 1 + v + v^2 + v^3 + \ldots$. We can see from these definitions that $\ddot{a}_{\overline{\infty}|i} - a_{\overline{\infty}|i} = 1$, and thus that $\ddot{a}_{\overline{\infty}|i} = 1 + a_{\overline{\infty}|i}$.

We now summarize what we have learned about perpetuities.

Perpetuities

- The present value of a **perpetuity immediate** that pays 1 at the end of each year, continuing forever, is given by $a_{\overline{\infty}|i} = v + v^2 + v^3 + \ldots = \dfrac{1}{i}$.
- The present value of a **perpetuity due** that pays 1 at the beginning of each year, continuing forever, is given by $\ddot{a}_{\overline{\infty}|i} = 1 + v + v^2 + \ldots = \dfrac{1}{d}$.
- The following identities can be derived from the definitions of perpetuities immediate and perpetuities due.
  - $\ddot{a}_{\overline{\infty}|i} = (1+i)a_{\overline{\infty}|i}$
  - $\ddot{a}_{\overline{\infty}|i} = 1 + a_{\overline{\infty}|i}$

Example 2.20

The following annuities all have the same present value.

a) A perpetuity due with annual payments of 100, at an annual effective interest rate of i.

b) A perpetuity immediate with annual payments of 135, at an annual effective interest rate of $1.25i$.

c) An n-year annuity immediate with annual payments of 148, at an annual effective interest rate of i.

Find n.

Non-Annual Payments

The formulas above can be applied to perpetuities with non-annual payments. As with annuities, we would simply replace the annual effective rate i with the effective periodic rate associated with the payment period.

Example 2.21

A perpetuity pays 2 at the end of each odd-numbered year, and 5 at the end of each even-numbered year. Find the present value of this perpetuity at $i = 10\%$.

Example 2.22

A perpetuity-immediate makes payments in the following sequence, forever: 1, 2, 3, 1, 2, 3, … . Find the present value of this perpetuity at $i = 10\%$.

The present value of the perpetuity in Example 2.22 can also be found using a technique called fusion. We will discuss this method in Section 2.5.

2.4 DEFERRED ANNUITIES AND BLOCK PAYMENTS

Deferred Annuities

You will occasionally encounter an annuity in which the first payment period does not begin at time $t = 0$. Such an annuity is called a **deferred annuity**. We will use the symbol ${}_{m|}a_{\overline{n}|i}$ to refer to the time $t = 0$ present value of a deferred annuity immediate that makes n annual payments of 1, with the first payment period beginning at time $t = m$. The first payment of this annuity would occur at time $t = m + 1$, and the last payment would occur at time $t = m + n$. A time diagram for such an annuity is pictured below.

0	1	2	•••	m	$m+1$	$m+2$	•••	$m+n-1$	$m+n$
			•••		1	1	•••	1	1

The are two commonly used approaches for calculating the present value ${}_{m|}a_{\overline{n}|i}$:

1. Calculate the present value of the n payments at time $t = m$. This present value would be equal to $a_{\overline{n}|i}$. We then discount this quantity by m years to get the present value at time $t = 0$. This results in the formula ${}_{m|}a_{\overline{n}|i} = v^m a_{\overline{n}|i}$.

2. Start by considering an annuity immediate that makes payments for $m + n$ years. The present value of this annuity would be $a_{\overline{m+n}|i}$. We then subtract from this quantity the present value of the first n "missing" payments. This provides us with the formula ${}_{m|}a_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{n}|i}$.

The second method mentioned above is a special case of the block payments method of calculating present value. We will discuss this method later in this section.

Deferred Annuities

The symbol ${}_{m|}a_{\overline{n}|i}$ represents the present value of an n-year annuity immediate that pays 1 at the end of each year, with the first payment period starting at time $t = m$. The first payment of this annuity would occur at time $t = m + 1$ and the last payment would occur at time $t = m + n$. We present two formulas for calculating ${}_{m|}a_{\overline{n}|i}$:

- ${}_{m|}a_{\overline{n}|i} = v^m a_{\overline{n}|i}$
- ${}_{m|}a_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{n}|i}$

Example 2.23 Deposits of 60 are made into a fund at the beginning of each year for 8 years with the first deposit occurring at time $t = 10$. The effective annual interest rate is 4%. Calculate the present value of this series of payments.

Example 2.24 A company is planning for two sequences of future payments that they are obligated to deliver.

- The first sequence consists of semi-annual payments of 500 lasting for 6 years, with the first payment taking place exactly 5 years from today.
- The second sequence consists of semi-annual payments of 750 lasting for 6 years, with the first payment taking place exactly 8 years from today.

To cover these payments, the company will make semi-annual deposits of K into an account earning a nominal annual rate of 6%, convertible semi-annually. The deposits will last for 5 years, with the first deposit taking place today. Find K.

Block Payments

The method of **block payments** provides a convenient tool for calculating the present and accumulated values of an annuity in which the payments vary over time, but are constant for certain periods of time. An example of this would be a 10-year annuity immediate that made annual payments of 100 for the first 4 years, payments of 150 for the next 3 years, and payments of 300 for the final 3 years. The formulas for working with block payments are described below.

Block Payments

- Let $t_1, t_2, \ldots, t_n$ be an increasing sequence of whole numbers.
- Consider an annuity immediate that makes annual payments as follows:
  - Payments of P_1 are made at the end of years 1 through t_1.
  - Payments of P_2 are made at the end of years $t_1 + 1$ through t_2.

    ...
  - Payments of P_n are made at the end of years $t_{n-1} + 1$ through t_n.
- For $i = 1, 2, 3, \ldots, n-1$, let $\Delta_i = P_{i+1} - P_i$.
- The present value of this annuity at time $t = 0$ is given by:
  $PV = P_n a_{\overline{t_n}|} - \Delta_{n-1} a_{\overline{t_{n-1}}|} - \Delta_{n-2} a_{\overline{t_{n-2}}|} - \ldots - \Delta_2 a_{\overline{t_2}|} - \Delta_1 a_{\overline{t_1}|}$
- The accumulated value of this annuity at time $t = t_n$ is given by:
  $AV = P_1 s_{\overline{t_n}|} + \Delta_1 a_{\overline{t_n - t_1}|} + \Delta_2 a_{\overline{t_n - t_2}|} + \ldots + \Delta_{n-1} a_{\overline{t_n - t_{n-1}}|}$

The formulas presented above for block payments are complicated and likely somewhat confusing without additional context. One should probably not spend time attempting to memorize these formulas. The idea underlying block payments is probably best explained through examples.

Example 2.25 An annuity immediate pays 2 during years 1 – 6, and pays 8 during years 7 – 10. If $i = 10\%$, find the present value of this annuity, as well as the accumulated value at the end of year 10.

Example 2.26 An annuity immediate pays 5 during years 1 – 2, pays 3 during years 3 – 4, pays 9 during years 5 – 6, and pays 7 during years 7 – 8. Assume an annual effective interest rate of 7%.

a) Find the present value of this annuity at time $t = 0$.
b) Find the accumulated value of this annuity at time $t = 8$.

Example 2.27 Deposits are made into an account at the end of each year for $3n$ years as follows:

- Deposits of 75 are made for the first n years.
- Deposits of 100 are made for the middle n years.
- Deposits of 125 are made for the final n years.

The accumulated value of the account at the end of $3n$ years is equal to 9609. The annual effective interest rate earned by the account is i. You are given that $(1 + i)^n = 2$. Find i.

Example 2.28 Kevin borrows 10,000 at an annual effective interest rate of 6% and agrees to repay it by making 30 annual payments with the first payment due in one year. The size of the payments is set to double after the first ten years. After making the tenth payment, Kevin is given the option of repaying the loan by making a final payment of K. This would result in Kevin paying an annual effective rate of 7% over the lifetime of the loan. Find K.

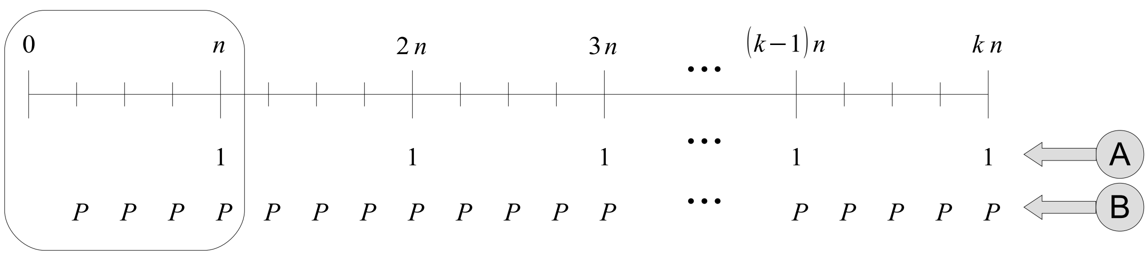
2.5 FUSION AND FISSION

There are times when it is convenient to convert a given annuity to one with equal present value, but with payments occurring either more or less frequently than the original annuity. The methods of Fission and Fusion allow us to "combine" or "split" payments in an annuity in order to achieve this goal.

Fission

The **fission** method is used to split annuity payments, creating a new annuity with more frequent payments. The process works as follows:

- Assume Annuity A makes k payments of 1 with payments occurring at the end of each n-year period. This is illustrated in the time diagram below with $n = 4$.
- We want to find an equivalent annuity (i.e. one with the same PV) that makes payments of P at the end of each year for the entire kn year period. Our goal is to find the appropriate value for P. We will call this new annuity Annuity B. It is also shown in the time diagram below.
- Notice that the first n payments in Annuity B must be equivalent to the payment of 1 at time n in Annuity A. It follows that $1 = P s_{\overline{n}|i}$ and $P = 1 / s_{\overline{n}|i}$.
- The PV of either annuity is thus $PV = P a_{\overline{kn}|i} = \dfrac{a_{\overline{kn}|i}}{s_{\overline{n}|i}}$.
- We conclude that the PV of an annuity paying 1 at the end of each n-year period for kn years is $\dfrac{a_{\overline{kn}|i}}{s_{\overline{n}|i}}$.



> **Fission**
>
> - The PV of an annuity paying 1 at the end of each n-year period for kn years is $\dfrac{a_{\overline{kn}|i}}{s_{\overline{n}|i}}$.

We could employ a similar strategy to show that the PV of an annuity paying 1 at the beginning of each n-year period for kn years is $\dfrac{a_{\overline{kn}|i}}{a_{\overline{n}|i}}$.

The discussion above explains how to convert an annuity that makes payments less frequently than annually into an annuity that pays annually. The same process could be used to convert any annuity into one that makes payments more frequently, regardless of what the actual periods are. For instance, you could use fission to convert an annual annuity into a monthly annuity.

You should be familiar with the the formula that we have derived above, as well as the process used to obtain it.

Converting Rates

There are many problems for which fission is a valid strategy, but that can be solved more easily by simply converting rates. For instance, assume you want to find the present value of an annuity that pays R every 3 years for 30 years. If you know the effective annual interest rate i, then you can find the effective 3 year rate j with $(1+i)^3 = 1+j$. The present value of the annuity could then be calculated using $PV = R\,a_{\overline{10}|j}$. The fission method is most useful when the interest rate is unknown, and thus cannot be converted. It is also useful for certain types of symbolic problems.

Example 2.29

An annuity pays 10 at the end of each 4 year period for 28 years. The effective annual rate of interest is $i = 6\%$.

a) Find the PV of this annuity by converting rates.
b) Find the PV of this annuity using fission.

Example 2.30

You are given the following information.

i. The present value of an n-year annuity immediate with payments of 1 at the end of every year is 16.4978.
ii. The present value of an n-year annuity immediate with payments of 1 at the end of every two years is 8.0322.
iii. The present value of an n-year annuity immediate with payments of 1 at the end of every three years is X.

Find X.

Example 2.31

Assume an annual effective interest rate of i. You are given the following:

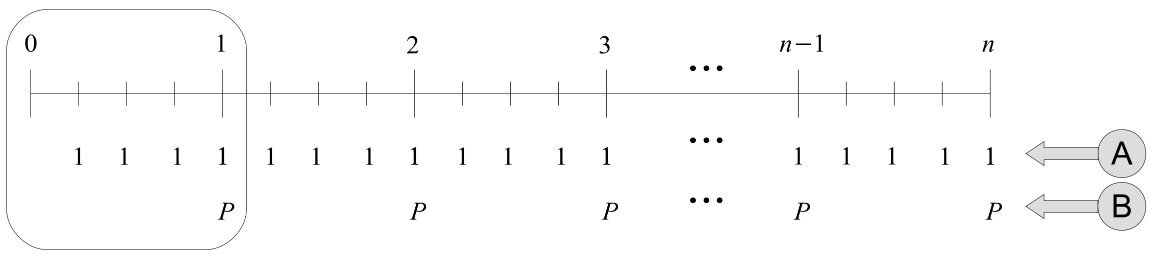
i. The present value of an annuity immediate paying 1 every n years for $4n$ years is equal to 2.6182.
ii. the present value of an annuity immediate paying 1 every $2n$ years for $4n$ years is equal to 1.1933.

Find the present value of an annuity immediate paying 1 at the end of each year for $4n$ years.

Fusion

Fusion is a method of combining annuity payments to create a new annuity with less frequent payments than the original one. A description of the method is provided below.

- Assume Annuity A makes m-thly payments of 1 for n years with payments occurring at the end of each $1/m$ year period. This is illustrated in the time diagram below with $m = 4$.
- We wish to find a second annuity, which we will call Annuity B, that makes payments of P at the end of each year for the entire n year period. Our goal is to find the appropriate value for P.
- Notice that the first m payments of 1 in Annuity A must be equivalent to the payment of P at time 1 in Annuity B. It follows that $P = s_{\overline{m}|j}$, where j is the effective m-thly rate.
- The PV of either annuity is thus $PV = P a_{\overline{n}|i} = \left(s_{\overline{m}|j}\right)\left(a_{\overline{n}|i}\right)$.



> **Fusion**
>
> - The PV of an annuity paying 1 at the end of each $(1/m)$-year period for n years is given by $PV = P a_{\overline{n}|i} = \left(s_{\overline{m}|j}\right)\left(a_{\overline{n}|i}\right)$ where j is the effective m-thly rate and i is the annual effective rate.

Since we are required to know j to use the fusion method, it is often unnecessary to use fusion to solve this sort of problem. If we already know j, then we could have calculated the present value of Annuity A by using the formula $PV = a_{\overline{nm}|j}$.

Fusion is most useful when the payments in the annuity vary in some sort of periodic manner. It is also common to see fusion employed in symbolic problems.

Example 2.32

You are given a perpetuity with annual payments as follows:

i) Payments of 2 at the end of the first year, and every three years thereafter.
ii) Payments of 7 at the end of the second year, and every three years thereafter.
iii) Payments of 4 at the end of the third year, and every three years thereafter.

The interest rate is 8% annual effective. Find the present value of this perpetuity.

Example 2.33

An annuity immediate makes n payments per year for 5 years. The size of the individual payments is equal to P during the first year, $2P$ during the second year, $3P$ during the third year, $4P$ during the fourth year, and $5P$ during the fifth year. The present value of the first n payments of P is equal to 140. Assuming an annual effective interest rate of 6%, find the present value of this annuity.

m-thly Paying Annuities

As an alternative to using fusion or converting rates, you may use the formulas detailed below to calculate the present value or accumulated value of an annuity that makes m-thly payments. It doesn't hurt to memorize these formulas, but it should not be a priority. Most problems of this type can be solved just as easily using other techniques you are familiar with. You should understand the notation used in these formulas, however.

- When an annuity symbol includes a superscript of (m), this indicates that the annuity pays out a total of 1 over the course of each year, but does so in m-thly installments of $1/m$.
- Assume an n-year annuity immediate makes m-thly payments of $1/m$.
  - The PV of this annuity is $a_{\overline{n}|}^{(m)} = \dfrac{1-v^n}{i^{(m)}}$.
  - The AV of this annuity is $s_{\overline{n}|}^{(m)} = \dfrac{(1+i)^n - 1}{i^{(m)}}$.
- Assume an n-year annuity due makes m-thly payments of $1/m$.
  - The PV of this annuity is $\ddot{a}_{\overline{n}|}^{(m)} = \dfrac{1-v^n}{d^{(m)}}$.
  - The AV of this annuity is $\ddot{s}_{\overline{n}|}^{(m)} = \dfrac{(1+i)^n - 1}{d^{(m)}}$.
- Assume a perpetuity immediate makes m-thly payments of $1/m$.
  - The PV of this perpetuity is $a_{\overline{\infty}|}^{(m)} = \dfrac{1}{i^{(m)}}$.
- Assume a perpetuity due makes m-thly payments of $1/m$.
  - The PV of this perpetuity is $\ddot{a}_{\overline{\infty}|}^{(m)} = \dfrac{1}{d^{(m)}}$.
- When dealing with an expression such as $R\,a_{\overline{n}|}^{(m)}$, it is important to remember that R represents the *total* of the payments made over the course of the year, and not the individual payments themselves.

Example 2.34

Which of the following statements are true?

(I) $s_{\overline{n}|} - a_{\overline{n}|} = i\,a_{\overline{n}|}s_{\overline{n}|}$

(II) $\ddot{s}_{\overline{n}|}^{(m)} - s_{\overline{n}|}^{(m)} = \dfrac{i^{(m)}}{m}s_{\overline{n}|}^{(m)}$

(III) ${}_{1/4|}\ddot{a}_{\overline{n}|}^{(2)} + a_{\overline{n}|}^{(2)} = a_{\overline{n}|}^{(4)}$

(A) I only (B) III only (C) I and II only
(D) II and III only (E) I , II, and III

2.6 ARITHMETIC ANNUITIES

Up to this point, we have primarily worked with level annuities in which the payments stay constant from one period to the next. In this section and the next, we will consider annuities in which the payments increase or decrease in some prescribed manner. We will consider annuities where the payments change arithmetically in this section, and in the next section we will consider annuities where the payments form a geometric sequence.

General Arithmetic Annuities (P/Q Formulas)

Consider an n-year annuity immediate in which the first payment is equal to P and each subsequent payment increases by a fixed amount Q. This is the general form for an arithmetic annuity. A time diagram for such an annuity is shown on the right.

0	1	2	3	$\cdots$	n
	P	$P+Q$	$P+2Q$	$\cdots$	$P+(n-1)Q$

Let A denote the present value of this annuity. We will now derive a formula for A.

1. Notice that $A = Pv + (P+Q)v^2 + (P+2Q)v^3 ... + (P+(n-1)Q)v^n$.
2. This expression can be rewritten as $A = P\left[v + v^2 + ... + v^n\right] + Q\left[v^2 + 2v^3 + ... + (n-1)v^n\right]$.
3. Since $a_{\overline{n}|} = v + v^2 + ... + v^n$, we see that $A = Pa_{\overline{n}|} + Q\left[v^2 + 2v^3 + ... + (n-1)v^n\right]$.
4. Let $X = v^2 + 2v^3 + ... + (n-1)v^n$. Then $A = Pa_{\overline{n}|} + QX$.
5. Notice that $(1+i)X = v + 2v^2 + ... + (n-1)v^{n-1}$.
6. It follows that $(1+i)X - X = v + v^2 + v^3 + ... + v^{n-1} - (n-1)v^n$.
7. This simplifies to $iX = v + v^2 + v^3 + ... + v^{n-1} + v^n - nv^n$.
8. If follows that $iX = a_{\overline{n}|} - nv^n$, and $X = \dfrac{a_{\overline{n}|} - nv^n}{i}$.
9. Thus, we see that $A = Pa_{\overline{n}|} + Q\dfrac{a_{\overline{n}|} - nv^n}{i}$.

Let S be the accumulated value of this arithmetic annuity at time $t = n$. A formula for S can be obtained using a method similar to that which we used above to find the formula for A. Alternately, we could multiply the expression we derived for A by $(1+i)^n$. Doing so would yield the formula $S = Ps_{\overline{n}|} + Q\left(s_{\overline{n}|} - n\right)/i$.

Arithmetic Annuities (P/Q Formulas)

Consider an n-year annuity immediate in which the first payment is equal to P , and each subsequent payment increases by a fixed amount Q .

- The present value of this annuity at time $t = 0$ is given by $A = Pa_{\overline{n}|} + Q\dfrac{a_{\overline{n}|} - nv^n}{i}$.
- The accumulated value of this annuity at time $t = n$ is given by $S = Ps_{\overline{n}|} + Q\dfrac{s_{\overline{n}|} - n}{i}$.

Example 2.35 At $i = 8\%$, find the present value and accumulated value at $t = 6$ for the 6-year annuities immediate whose payments are given by each of the following sequences:

a) 12, 14, 16, 18, 20, 22

b) 20, 17, 14, 11, 8, 5, 2

Calculator Tip: An expression such as $S = 12 s_{\overline{6}|8\%} + 2\dfrac{s_{\overline{6}|8\%} - 6}{0.08}$ can be quickly calculated using the TI-30X by entering the following two commands: ▶ $\dfrac{1.08^6 - 1}{0.08} \to x$ ▶ $12 * x + 2 * \dfrac{x - 6}{0.08}$

Example 2.36

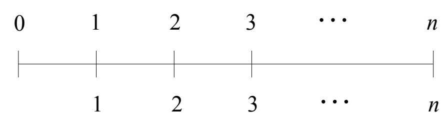
Assuming an annual effective interest rate of 7%, the following annuities have the same PV:

- An annuity immediate making quarterly payments of R for 10 years.
- An increasing annuity immediate with 10 annual payments, with the first payment in the amount of 400, and with subsequent payments increasing by 50 each year.

Find R.

Standard Increasing Annuities

Consider the special case of an arithmetic annuity in which $P = Q = 1$. This is an increasing annuity in which the payment at the end of any given year is equal to the number of years that have passed. A time diagram for such an annuity is shown on the right.



Annuities such as this are encountered frequently enough that we will introduce special notation for working with them. We will denote the present value of such an annuity by $(Ia)_{\overline{n}|}$ and will let $(Is)_{\overline{n}|}$ refer to the accumulated value of the annuity. By making the substitutions $P = Q = 1$ into the general formulas for arithmetic annuities and then simplifying, we obtain the following formulas.

Standard Increasing Annuities

Consider an n-year annuity immediate whose payments follow the sequence 1, 2, 3, …, n.

- The present value of this annuity at time $t = 0$ is given by $(Ia)_{\overline{n}|} = \dfrac{\ddot{a}_{\overline{n}|} - n v^n}{i}$.
- The accumulated value of this annuity at time $t = n$ is given by $(Is)_{\overline{n}|} = \dfrac{\ddot{s}_{\overline{n}|} - n}{i}$.

Notice that if an annuity immediate makes payments following the sequence R, $2R$, $3R$, …, nR, then its present and accumulated values are given by $R(Ia)_{\overline{n}|}$ and $R(Is)_{\overline{n}|}$, respectively.

Example 2.37

At $i = 7\%$, find the present value of an 12-year annuity immediate whose payments are given by the following sequence: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48.

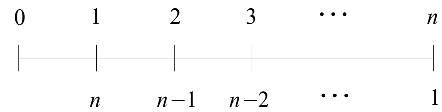
Calculator Tip: The value of an expression such as $4(Ia)_{\overline{12}|7\%}$ can be quickly calculated using the TI-30X by entering the following two commands: ▶ $\dfrac{1 - 1.07^{-12}}{0.07} \to x$ ▶ $4 * \dfrac{1.07 * x - 12 * 1.07^{-12}}{0.07}$

Example 2.38

Julia makes deposits at the end of each year into an account earning an annual effective interest rate of i. The first deposit is equal to 200, and subsequent deposits increase by 200 each year. The amount of interest earned by Julia's account during the tenth year is equal to 800. Find i.

Standard Decreasing Annuities

We now consider the special case of an arithmetic annuity in which $P = n$ and $Q = -1$. This results in a decreasing n-year annuity that pays n at $t=1$, with payments decreasing by 1 each year until reaching a final payment of 1 at $t=n$. A time diagram for such an annuity is shown on the right.



We will denote the present value of such an annuity with $(Da)_{\overline{n}|}$ and will denote the accumulated value by $(Ds)_{\overline{n}|}$. We obtain formulas for these values by substituting $P = n$ and $Q = -1$ into the general formulas for arithmetic annuities and then simplifying.

Standard Decreasing Annuities

Consider an n-year annuity immediate whose payments follow the sequence n, $n - 1$, …, 3, 2, 1.

- The present value of this annuity at time $t = 0$ is given by $(Da)_{\overline{n}|} = \dfrac{n - a_{\overline{n}|}}{i}$.
- The accumulated value of this annuity at time $t = n$ is given by $(Ds)_{\overline{n}|} = \dfrac{n(1+i)^n - s_{\overline{n}|}}{i}$.

If an annuity immediate makes payments following the sequence nR, $(n\text{-}1)R$, …, $3R$, $2R$, R, then its present and accumulated values are given by $R(Da)_{\overline{n}|}$ and $R(Ds)_{\overline{n}|}$, respectively.

Example 2.39

At i = 6%, find the present value of a 15-year annuity immediate whose payments are given by the following sequence: 45, 42, 39, 36, 33, 30, 27, 24, 21, 18, 15, 12, 9, 6, 3.

Calculator Tip: The value of an expression such as $3(Da)_{\overline{15}|6\%}$ can be quickly calculated using the TI-30X by entering the following two commands: ▶ $\dfrac{1 - 1.06^{-15}}{0.06} \to x$ ▶ $3 * \dfrac{15 - x}{0.06}$

Example 2.40

At an annual effective interest rate of 8%, the following annuities have the same present value:

- A 12-year annuity immediate that pays $50t$ at the end of year t.
- A 12-year annuity immediate that pays $P(13 - t)$ at the end of year t.

Find P.

Non-Annual Annuities with Annual Increases

Assume you have an annuity that makes m-thly payments for $m > 1$. Further assume that the payments are K during the entire first year, but increase by an amount of D at the end of each year, to remain constant for another full year. The present or accumulated value of such an annuity can be calculated using fusion. We fuse together the first years worth of payments to obtain an initial annual fused payment of $P = K\, s_{\overline{m}|j}$. We then fuse together the increases in the payments to obtain an annual fused increase of $Q = D\, s_{\overline{m}|j}$. We can then apply the P/Q formulas to calculate the present and accumulated values of this annuity.

Example 2.41 An 8-year annuity immediate makes quarterly payments. The payments are 5 per quarter during the first year, 6 per quarter during the second year, 7 per quarter during the third year, and so on, ending at 12 per quarter in the eighth year. Assuming a nominal annual rate of 6% convertible quarterly, find the present value of this annuity.

Increasing Perpetuities

Three types of increasing perpetuities are introduced in the box below. The derivations for the present values of these annuities has been omitted.

Increasing Perpetuities

- **General Increasing Perpetuities:** Consider a perpetuity immediate that pays P at the end of the first year with subsequent payments increasing by Q per year. The present value of this perpetuity is given by $PV = \frac{P}{i} + \frac{Q}{i^2}$.
- **Standard Increasing Perpetuities:** Assume a perpetuity immediate pays 1 at the end of the first year with subsequent payments increasing by 1 per year. The present value of such a perpetuity is denoted by $(Ia)_{\overline{\infty}|}$ and is given by $(Ia)_{\overline{\infty}|} = \frac{1}{i\,d} = \frac{1}{i} + \frac{1}{i^2}$.
- **Increasing to Level Perpetuities**: Consider a perpetuity immediate paying 1, 2, 3, …, n for the first n years and paying n at the end of each subsequent year. The present value of this perpetuity is $PV = \frac{\ddot{a}_{\overline{n}|i}}{i}$.

Example 2.42 Find the present value of a perpetuity immediate that pays 9 at the end of the first year, with each subsequent payment increasing by 4. Assume an annual effective interest rate of 5%.

Example 2.43 Find the present value of a perpetuity-immediate with payments starting at 5, increasing by 5 each year until reaching 100, and then staying at 100 from then on. Assume that $i = 8\%$.

m-thly Increasing Annuities

We will consider two types of *m*-thly increasing annuities. The first type makes *m*-thly payments, but the payments increase only at the end of each year. The second type makes *m*-thly payments, with the increases also occurring on an *m*-thly basis. You should be familiar with the notation presented here, if not the actual formulas.

- The symbol $(Ia)^{(m)}_{\overline{n}|}$ represents the PV of an annuity immediate that makes level *m*-thly payments of k / m during year *k*. In other words, the annuity makes level *m*-thly payments of $1 / m$ during year 1, level *m*-thly payments of $2 / m$ during year 2, and so on. It can be shown that $(Ia)^{(m)}_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n v^n}{i^{(m)}}$.
- The symbol $(I^{(m)}a)^{(m)}_{\overline{n}|}$ represents the PV of an annuity immediate that makes *m*-thly payments, with the first payment equal to $1/m^2$, and with *each* subsequent payment increasing by $1/m^2$. It can be shown that $(I^{(m)}a)^{(m)}_{\overline{n}|} = \frac{\ddot{a}^{(m)}_{\overline{n}|} - n v^n}{i^{(m)}}$.

2.7 GEOMETRIC ANNUITIES

Geometric Annuities

In this section we will consider annuities in which the payments form a geometric, rather than arithmetic, sequence. In other words, the ratio between any two consecutive payments will be a constant.

Consider an n-year annuity-immediate in which the first payment is P, and each subsequent payment increases by a factor of $(1+k)$. That is to say, each payment is $k \times 100$ % larger than the previous. The time diagram for such an annuity is shown below.

0	1	2	3	$\cdots$	n
	P	$P(1+k)$	$P(1+k)^2$	$\cdots$	$P(1+k)^{n-1}$

The present value of this annuity would be given by the following expression:

$$PV \;=\; Pv + P(1+k)v^2 + P(1+k)^2 v^3 + P(1+k)^3 v^4 + \ldots + P(1+k)^{n-1} v^n$$

The right-hand side of this equation can be rewritten as follows:

$$PV \;=\; \frac{P}{1+k}\left[(1+k)v + P(1+k)^2 v^2 + P(1+k)^3 v^3 + P(1+k)^4 v^4 + \ldots + P(1+k)^n v^n\right]$$

Let $v' = (1+k)v$. Substituting this value into the equation above gives us:

$$PV \;=\; \frac{P}{1+k}\left[v' + (v')^2 + (v')^3 + \ldots + (v')^n\right]$$

We could write the expression $\left[v' + (v')^2 + (v')^3 + \ldots + (v')^n\right]$ as $a_{\overline{n|}i'}$ if we could find an appropriate interest rate i'. In particular, we would need to find a rate i' such that $v' = \dfrac{1}{1+i'}$. Since $v' = (1+k)v = \dfrac{1+k}{1+i}$, this gives us $\dfrac{1}{1+i'} = \dfrac{1+k}{1+i}$. Solving this equation for i' gives us $i' = \dfrac{i-k}{1+k}$.

Thus, we conclude that the present value of the geometric annuity is given by $PV = \dfrac{P}{1+k} a_{\overline{n|}i'}$ where $i' = \dfrac{i-k}{1+k}$.

The recommended method of finding the accumulated value of this annuity is to first find the present value, and then accumulate this forward to time n. Note that you will need to use the true effective interest rate i, and not i', to accumulate the annuity forward.

Geometric Annuities

Consider an n-year annuity-immediate in which the first payment is P and each subsequent payment increases by a factor of $(1+k)$.

- The present value of this annuity at $t=0$ is given by $PV = \dfrac{P}{1+k} \cdot a_{\overline{n|}i'}$ where $i' = \dfrac{i-k}{1+k}$.
- The accumulated value of this annuity at $t=n$ is given by $AV = \dfrac{P}{1+k} \cdot a_{\overline{n|}i'}(1+i)^n$

Example 2.44 Rusty purchases a 20-year annuity immediate. The first annuity payment is 1000 and the payments increase by 2% each year. Assuming an annual effective interest rate of 5%, find the price Rusty paid for this annuity.

It is also possible for payments in an annuity to decrease at a geometric rate. This results in a negative value for k.

Example 2.45 The first payment in an annuity immediate is equal to 2500. Each subsequent payment is 3% less than the previous payment. The payments continue for as long as their value is greater than 1500. At an annual effective interest rate of 6%, find the present value of this annuity.

As with arithmetically increasing annuities, we will use fusion to work with m-thly annuities for which the payments remain constant throughout the year, but increase in a geometric fashion at the end of each year.

Example 2.46 A loan of 275,000 will be repaid by monthly payments over the course of 20 years. The first payment is in the amount of P and occurs one month after the loan was made. Payments are level during any given year, but increase by 2.5% at the end of each year. Assuming an annual effective interest rate of 6%, find P.

Geometric Perpetuities

The special interest rate i' can be used to work with geometrically perpetuities as well as geometric annuities.

Geometric Perpetuities

Assume that a the first payment in an perpetuity immediate is equal to P, and each subsequent payment increases by a factor of $(1+k)$.

- The present value of this perpetuity at $t=0$ is given by $PV = \dfrac{P}{1+k} \cdot a_{\overline{\infty}|i'}$ where $i' = \dfrac{i-k}{1+k}$.

Example 2.47 A perpetuity immediate makes level payments of 100 for 8 years. After the first 8 years, each payment is 4% greater than the previous payment. Assuming an annual effective interest rate of 9%, calculate the present value of this perpetuity.

So far, we have only encountered examples in which the rate of increase in payments is strictly less than the annual effective interest rate. Notice that if $k = i$, then $i' = 0$, which results in $a_{\overline{n}|i'} = n$. If $k > i$, then $i' < 0$. The formulas provided will still work in this case. Just make sure to pay careful attention to the sign on i' when entering the rate into an annuity formula or into the BA II Plus calculator.

Example 2.48 An annuity immediate makes annual payments for 10 years. The first payment is equal to 50, and each subsequent payment is K% higher than the previous payment. Assuming an annual effective interest rate of 4%, calculate the present value of the annuity in each of the following cases: (a) $K = 2$, (b) $K = 4$, (c) $K = 6$

2.8 CONTINUOUS ANNUITIES

In this section, we will consider annuities which make payments continuously. Such annuities do not literally exist, but can serve as useful approximations for funds that make very frequent payments. To calculate the present value of such an annuity, we will need to know the rate at which the payments are made, as well as the continuous force of interest.

General Formulas for Continuous Paying Annuities

Assume that an annuity makes continuous payments to the owner of the annuity. The payments continue for n years and the rate of payment at time t is given by $f(t)$. Assume that interest is accumulated at a continuously changing force of interest δ_t which is associated with an accumulation function $a(t)$.

Consider an infinitesimal interval of time centered at time t and with length dt. The amount of money payed during this interval of time is $f(t)dt$. The present value of the payment $f(t)dt$ is equal to $f(t)dt / a(t)$. Summing the present values of all such payments over all possible times t results in the following integral for the total present value of the annuity: $PV = \int_0^n f(t) / a(t) dt$. We can derive a similar integral for the accumulated value of such an annuity, but it would be more complicated. It is generally simpler to calculate the present value and then accumulate that forward to find the accumulated value.

General Formulas for Continuous Paying Annuities

Assume an n-year annuity makes continuous payments at a rate of $f(t)$ at time t. Assume that interest is accumulated at a force of interest δ_t which results in an accumulation function $a(t)$.

- The present value of this annuity at $t = 0$ is given by $PV = \int_0^n \frac{f(t)}{a(t)} dt$.
- The accumulated value of this annuity at $t = n$ is given by $AV = a(n) \cdot PV$.

Example 2.49 An 8-year annuity makes continuous payments at a rate of $5t$ at time t. Assume a continuous force of interest given by $\delta_t = 0.2t / (1 + 0.1t^2)$.

a) Find the present value of this annuity.

b) Find the accumulated value of this annuity at time $t = 8$.

Example 2.50 Payments are made to an account at a continuous rate of $(6k + 6k)$. Assume a continuous force of interest given by $\delta_t = 1 / (8 + t)$. And the end of the fifth year, the account is worth 12,000. Find k.

Example 2.51 An annuity makes continuous payments at a rate $6t^2$ for 10 years. The price of this annuity is determined using a constant force of interest $\delta = 0.10$. Find the price of the annuity.

We will now consider several special cases of continuous-paying annuities.

Constant Payments and Constant Force of Interest

Consider an annuity that makes continuous payments at a constant rate of $f(t) = 1$ per year. Assume that the force of interest is given by a constant δ. The present value of such an annuity is denoted by the symbol $\bar{a}_{\overline{n}|}$ and the accumulated value is denoted by $\bar{s}_{\overline{n}|}$.

Since δ is constant, we are working with compound interest and $a(t) = e^{\delta t} = (1+i)^t$, where i is the associated annual effective rate of interest. We also note that since we are working with compound interest, the present value factor $1 / a(t)$ can be written as $1 / a(t) = v^t = e^{-\delta t} = (1+i)^{-t}$.

Substituting $f(t) = 1$ and $1 / a(t) = v^t$ into the general present value formula for a continuous-paying annuity yields the integral $\bar{a}_{\overline{n}|} = \int_0^n v^t \, dt$. Solving this integral provides us with the formula $\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}$.

By accumulating this expression forward, we see that the accumulated value of this annuity is $\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$.

Continuous Paying Annuity with Constant Payments and Constant Force of Interest

Assume an n-year annuity makes continuous payments at a rate of 1 per year. Assume that interest is accumulated at a constant force of interest δ.

- The present value of this annuity at $t = 0$ is given by $\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}$.
- The accumulated value of this annuity at $t = n$ is given by $\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$.

Notice that the formulas for $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$ are very similar to those for $a_{\overline{n}|}$ and $s_{\overline{n}|}$. The only difference is that the i in the denominator of the standard annuity formulas is replaced with δ for continuous-paying annuities.

If an annuity pays R each year, paid continuously over the course of the year, then its present and accumulated values are given by $R\bar{a}_{\overline{n}|}$ and $R\bar{s}_{\overline{n}|}$, respectively.

Example 2.52

A 20-year annuity makes continuous payments at a rate of 8 per year. Assume $i = 5\%$.

a) Find the present value of this annuity at $t = 0$.
b) Find the accumulated value of this annuity at $t = 20$.

Example 2.53

A 12-year annuity makes continuous payments at a rate of 3 per year. In addition to the continuous payments, discrete payments of 2 are made at the end of each year. The effective annual rate of interest is 6%.

a) Find the present value of this annuity at $t = 0$.
b) Find the accumulated value of this annuity at $t = 12$.

Example 2.54

You are given that $\bar{a}_{\overline{12}|} = 6.988$ and $\frac{d}{d\delta}\left(\bar{a}_{\overline{10}|}\right) = -33.737$. Find δ.

Notice that the equation $\bar{a}_{\overline{12}|} = 6.988$ in the previous example has only one unknown in it: δ. This problem can be solved quickly by using the Table function in the TI-30X to plug several different values of δ into $\bar{a}_{\overline{12}|}$.

Increasing Continuous Annuities

We will now consider the special case of a continuous-paying annuity in which the rate of payment increases linearly over time and the force of interest is constant. To that end, assume that $f(t) = t$ and let δ denote the constant force of interest. The present value of such an annuity is denoted by $(\bar{I}\,\bar{a})_{\overline{n|}}$ and its accumulated value is denoted by $(\bar{I}\,\bar{s})_{\overline{n|}}$.

Substituting $f(t) = t$ and $1 / a(t) = v^t$ into the general present value formula for a continuous-paying annuity yields the integral $(\bar{I}\,\bar{a})_{\overline{n|}} = \int_0^n t\,v^t\,dt$. Solving this integral gives us $(\bar{I}\,\bar{a})_{\overline{n|}} = \dfrac{\bar{a}_{\overline{n|}} - n\,v^n}{\delta}$.

By accumulating this expression forward, we see that the accumulated value of this annuity is $(\bar{I}\,\bar{s})_{\overline{n|}} = \dfrac{\bar{s}_{\overline{n|}} - n}{\delta}$.

> **Continuous Paying Annuity with Increasing Payments and Constant Force of Interest**
>
> Assume an *n*-year annuity makes continuous payments at a rate of $f(t) = t$ and interest is accumulated at a constant force of interest δ .
>
> - The present value of this annuity at $t = 0$ is given by $(\bar{I}\,\bar{a})_{\overline{n|}} = \dfrac{\bar{a}_{\overline{n|}} - n\,v^n}{\delta}$.
> - The accumulated value of this annuity at $t = n$ is given by $(\bar{I}\,\bar{s})_{\overline{n|}} = \dfrac{\bar{s}_{\overline{n|}} - n}{\delta}$.

If an annuity makes continuous payments at a rate of $f(t) = k\,t$, then its present and accumulated values are given by $k(\bar{I}\,\bar{a})_{\overline{n|}}$ and $k(\bar{I}\,\bar{s})_{\overline{n|}}$, respectively.

Example 2.55

A 12-year annuity makes continuous payments at a rate of 4*t*. Assume $i = 6\%$.

a) Find the present value of this annuity at $t = 0$.

b) Find the accumulated value of this annuity at $t = 12$.

Example 2.56

At an annual effective rate of 7%, the following annuities have the same present value:

i) A 10-year annuity that makes continuous payments at a constant rate of 8 per year.

ii) A 10-year annuity that makes continuous payments at a rate of $k\,t$ at time *t*.

Find *k*.

Decreasing Continuous Annuities

Next we will look at continuous-paying annuities in which the rate of payment decreases linearly over time and the force of interest is constant. Assume that $f(t) = n - t$ and let δ denote the constant force of interest. The present value of such an annuity is denoted by $(\bar{D}\bar{a})_{\overline{n|}}$ and its accumulated value is denoted by $(\bar{D}\bar{s})_{\overline{n|}}$.

Substituting $f(t) = n - t$ and $1 / a(t) = v^t$ into the general present value formula for a continuous-paying annuity yields the integral $(\bar{D}\bar{a})_{\overline{n|}} = \int_0^n (n-t)v^t\,dt$. Solving this integral results in the formula $(\bar{D}\bar{a})_{\overline{n|}} = \dfrac{n - \bar{a}_{\overline{n|}}}{\delta}$. Accumulating this expression forward, gives us that $(\bar{D}\bar{s})_{\overline{n|}} = \dfrac{n(1+i)^n - \bar{s}_{\overline{n|}}}{\delta}$.

Continuous Paying Annuity with Decreasing Payments and Constant Force of Interest

Assume an n-year annuity makes continuous payments at a rate of $f(t) = n - t$ and interest is accumulated at a constant force of interest δ.

- The present value of this annuity at $t = 0$ is given by $(\bar{D}\bar{a})_{\overline{n|}} = \dfrac{n - \bar{a}_{\overline{n|}}}{\delta}$.
- The accumulated value of this annuity at $t = n$ is given by $(\bar{D}\bar{s})_{\overline{n|}} = \dfrac{n(1+i)^n - \bar{s}_{\overline{n|}}}{\delta}$.

If an annuity makes continuous payments at a rate of $f(t) = k(n-t)$, then its present and accumulated values are given by $k(\bar{D}\bar{a})_{\overline{n|}}$ and $k(\bar{D}\bar{s})_{\overline{n|}}$, respectively.

Example 2.57 A 15-year annuity makes continuous payments. The rate of payment is equal to 60 at $t = 0$, and decreases linearly until it reaches 0 at $t = 15$. The annual effective interest rate is 8%.

a) Find the present value of this annuity at $t = 0$.
b) Find the accumulated value of this annuity at $t = 15$.

Example 2.58 An annuity makes continuous payments for 30 years. Payments are made at a constant rate of 50 per year for the first 20 years. During the last 10 years, the rate of payment decreases linearly from 50 to 0. Find the present value of this annuity at an annual effective rate of interest is 7%.

2.9 MISCELLANEOUS ANNUITY TOPICS

We will conclude this chapter with a discussion of some miscellaneous topics related to annuities.

The $a_{\overline{kn}|} / a_{\overline{n}|}$ Formula

Notice that $a_{\overline{2n}|} = v + v^2 + \ldots + v^n + v^{n+1} + v^{n+2} + \ldots + v^{2n} = \left[v + v^2 + \ldots + v^n\right] + v^n\left[v + v^2 + \ldots + v^n\right] = a_{\overline{n}|}\left(1+v^n\right)$.
It follows that $a_{\overline{2n}|} / a_{\overline{n}|} = 1 + v^n$. Similar formulas can be derived for $a_{\overline{kn}|} / a_{\overline{n}|}$ where $k > 2$.

The $a_{\overline{kn}|} / a_{\overline{n}|}$ Formulas

The following identities hold for all values of n and i.

- $\dfrac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n$
- $\dfrac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n}$
- $\dfrac{a_{\overline{4n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n} + v^{3n}$

These formulas can be used to simplify the algebra involved in certain types of problems.

Example 2.59

At an annual effective interest rate i, the following annuities have the same present value.

i. A 12-year annuity immediate with level annual payments of 27.
ii. A 24-year annuity immediate with level annual payments of 20.

Find i.

Example 2.60

Assuming an annual effective interest rate i, you are given:

i. The present value of an n-year annuity due with annual payments of 100 is 1030.61.
ii. The present value of a $2n$-year annuity due with annual payments of 150 is 2099.24.

Find i.

Example 2.61

You are given $\ddot{a}_{\overline{n}|i} = 15.2191$ and $\ddot{a}_{\overline{2n}|i} = 21.1663$. Find n.

Example 2.62

An annuity immediate pays X per year for $3n$ years. The present value of the annuity is 1000. The present value of the first n payments is equal to 640. Find the present value of the last n payments.

Example 2.63

Assuming an annual effective interest rate i, you are given the following information:

i. The present value of an $2n$-year annuity immediate that pays 4 for the first n years and 3 for the last n years is 34.8113.
ii. The present value of an n-year deferred annuity immediate that pays 3 for n years is equal to 6.9728

Find i.

Selling Annuities and Early Repayment

Assume an individual purchases an n-year annuity immediate that pays R per year at an annual effective interest rate of i. Let P denote the price paid by the individual. In other words, P is the present value of the annuity at the annual effective rate of i.

Now assume that after receiving the first m payments, the individual sells the rights to receive the remaining $n-m$ payments to another party for a price of Q. Depending on the value of Q, the original owner of the annuity and the new purchaser could each realize interest rates that are different from each other, and both different from i. Let j denote the rate realized by the original owner of the annuity and let k denote the rate earned by the purchaser of the final $n-m$ payments. These rates can be determined as follows:

- The rate for the original owner is determined by the equation $P = R\,a_{\overline{m}|j} + Q v^{-m}$.
- The rate for the new purchaser is determined by the equation $Q = R\,a_{\overline{n-m}|k}$.

For most values of m, it is practically impossible to solve for the rate in an expression such as $P = R\,a_{\overline{m}|j} + Q v^{-m}$ without using some form of technology. Fortunately, the TVM feature of the BA II Plus calculator can be used to solve such problems.

As an example, assume that we wish to solve for j in the equation $100 = 12\,a_{\overline{8}|j} + 90\,v^{8}$. We could do so by entering the following information into the BA II Plus:

- [2ND] [CLR TVM] 8 [N] 100 [PV] 12 [+/-] [PMT] 90 [+/-] [FV] [CPT] [I/Y]

Example 2.64

Arthur borrows 2000 from Betty. Arthur agrees to repay the loan over the course of 10 years by making annual payments at an annual effective interest rate of 7%.

Immediately after receiving the sixth payment from Arthur, Betty sells the rights to receive the remaining four payments to Chad for a price of 1000. The size of the payments do not change, but Arthur now pays them to Chad.

a) Determine annual effective rate of interest earned by Chad.
b) Determine annual effective rate of interest earned by Betty.
c) Determine annual effective rate of interest paid by Arthur.

The approach above can also be used to solve for the interest rate in problems involving early repayment of a loan.

Example 2.65

Jorge borrows 160,000 at a nominal annual rate of interest of 4.8% convertible monthly. He agrees to repay the loan by making monthly payments of R.

At the end of the 15$^{\text{th}}$ year, immediately after making payment number 180, Jorge repays the remaining balance of the loan. He is also charged an early repayment penalty of 8000 at this time. Find the rate of interest actually paid by Jorge. Express your answer as a nominal annual rate of interest compounded monthly.

Varying Rates for Annuities

It is possible to encounter an annuity problem in which the effective rate of interest changes at some point during the duration of the annuity. In calculating the present or accumulated value of such an annuity, you will generally need to split the annuity into separate annuities such that the rate of interest is consistent within the time period spanned by each annuity. Be careful about using the correct rate of interest when discounting or accumulating through any particular time period.

For example, assume that we wish to find the present value of a 15-year annuity immediate with payments of 1 assuming an annual effective rate of 4% during years 1 – 10 and an annual effective rate of 6% during years 11 – 15. We can calculate the present value as follows:

1. Discount payments 11 – 15 to time 10 using $a_{\overline{5}|6\%}$.
2. Discount $a_{\overline{5}|6\%}$ through years 1 – 10 to time 0 by multiplying by $(1.04)^{-10}$.
3. Discount payments 1 – 10 to time 0 using $a_{\overline{10}|4\%}$. Add this to the result from Step 2.

The resulting present value is: $PV = a_{\overline{10}|4\%} + (1.04)^{-10} a_{\overline{5}|6\%}$.

Example 2.66

Anita deposits 60 into a fund at the end of each year for 20 years. The fund earns an annual effective interest rate of 5% for the first 14 years and an annual effective interest rate of 8% during the last 6 years.

a) Find the accumulated value of the fund at the end of the 20 years.
b) Find the overall annual effective rate of interest realized by Anita during the 20 year period.

Continuous Force of Interest, but Discrete Payments

Assume that you are asked to find the present value or accumulated value of an annuity that makes discrete payments, but you are given a continuous force of interest, δ_t. In this scenario, you can use the following formulas for $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

- $a_{\overline{n}|} = \dfrac{1}{a(1)} + \dfrac{1}{a(2)} + \dfrac{1}{a(3)} + \ldots + \dfrac{1}{a(n)}$
- $s_{\overline{n}|} = a(n)\left[\dfrac{1}{a(1)} + \dfrac{1}{a(2)} + \dfrac{1}{a(3)} + \ldots + \dfrac{1}{a(n)}\right]$

Example 2.67

Assume interest is credited according the the force of interest $\delta_t = \dfrac{2}{5+t}$, find $a_{\overline{4}|}$ and $s_{\overline{4}|}$.

Example 2.68

Erin makes deposits of K into an account at the end of each year for for 5 years. The account earns interest at a force of interest given by $\delta_t = \dfrac{0.2t}{1+0.01t^2}$. Determine the annual effective rate of interest that Erin earned over the course of the 5 years.

Palindromic Annuities

Consider an annuity immediate that makes annual payments that start at 1, increasing by 1 each year until they reach n, and then decreasing by 1 each year until they again reach 1. The payments in this annuity follow a palindromic pattern. The present value of this annuity can be calculated by treating it as an n-year increasing annuity plus an $(n-1)$-year decreasing annuity that is deferred by n years. Such an approach would yield a present value of $PV = (Ia)_{\overline{n}|} + v^n (Da)_{\overline{n-1}|}$. We will consider another approach to calculating the present value of such an annuity.

We first split the annuity into n level annuities, each of which pays 1 for n years. The time of the first payment for these n annuities will vary from 1 to n. You should convince yourself that this produces the same total payment at the end of each year as the original annuity.

We now calculate the present value of each sub-annuity at the time of its own first payment. This produces a set of n payments of $\ddot{a}_{\overline{n}|}$ at times 1, 2, …, n. Calculating the total present value of these payments results in the formula $PV = a_{\overline{n}|} \cdot \ddot{a}_{\overline{n}|}$.

We can take a similar approach to calculate the present value of a palindromic annuity that makes two consecutive payments of n before beginning to decrease. Time diagrams both types of palindromic annuities are shown below, along with formulas for their present values.

- $1 \quad 2 \quad 3 \quad \cdots \quad n-1 \quad n \quad n-1 \quad \cdots \quad 3 \quad 2 \quad 1$ $\qquad PV = a_{\overline{n}|} \cdot \ddot{a}_{\overline{n}|} = (1+i)\left[a_{\overline{n}|}\right]^2$

- $1 \quad 2 \quad 3 \quad \cdots \quad n-1 \quad n \quad n \quad n-1 \quad \cdots \quad 3 \quad 2 \quad 1$ $\qquad PV = a_{\overline{n}|} \cdot \ddot{a}_{\overline{n+1}|} = (1+i)\left[a_{\overline{n}|} a_{\overline{n+1}|}\right]$

Example 2.69

Assuming an annual effective interest rate of 5%, calculate the present value of each of the annuities described below.

a) An 11-year annuity immediate with payments of 1, 2, 3, 4, 5, 6, 6, 4, 3, 2, 1.
b) A 12-year annuity immediate with payments of 1, 2, 3, 4, 5, 6, 6, 5, 4, 3, 2, 1.
c) An 11-year annuity due with payments of 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1.
d) A 12-year annuity due with payments of 1, 2, 3, 4, 5, 6, 6, 5, 4, 3, 2, 1.

Example 2.70

At an annual effective interest rate of i, the following annuities have the same present value:

i) A 9-year annuity-due with payments of 1, 2, 3, 4, 5, 4, 3, 2, 1.
ii) An 8-year annuity-immediate with payments of k, $2k$, $3k$, $4k$, $4k$, $3k$, $2k$, k, where $k = 1.4$

CHAPTER 3 – Reinvestment and Amortization

3.1 REINVESTMENT

Effective Yield of an Annuity Without Reinvestment

Assume a loan of L accumulates interest at an annual effective interest rate of i and is to be repaid over the course of n years. Let's consider two possible methods by which the borrower could repay the loan:

1. The borrower could make a single lump sum payment of $L(1+i)^n$ at the end of n years.
2. The borrower makes annual payments of R at the end of each year. The size of these payments can be determined using the formula $L = R a_{\overline{n}|i}$.

The total amount that the lender receives during the n-year period is smaller in the second repayment plan than in the first. However, under the second payment plan, the lender will receive payments throughout the lifetime of the loan and can thus use this money to pursue other investments. If, however, the lender does not opt to reinvest the payments as they are received, then those payments will fail to earn interest after they are received. As a result, the lender's overall yield rate during the n-year period will be smaller than i. Let's illustrate this concept with an example.

- Assume you pay ABC Investments 421.24 to purchase a 5-year annuity-immediate with payments of 100.
- Since $421.24 = 100a_{\overline{5}|6\%}$, your yield on this investment is 6%.
- Notice that the sum of the payments you received is 500. However, the accumulated value of 421.24 invested for 5 years at 6% is $421.24(1.06)^5 = 563.71$, which is clearly greater than 500.
- As ABC makes payments to you, it no longer has to pay interest on the balance of those payments, which explains the apparent discrepancy in the previous bullet point. If no payments are made to you during the 5 years, then you would earn 6% interest on the entire balance of 421.25 for the 5 year period, thus yielding 563.71.
- Assume that we adopted an investment strategy of purchasing this annuity from ABC Investments and depositing the annuity payments under a mattress as they arrive. Our initial investment would be 421.24, and we would "cash out" for 500 at the end of 5 years. To find the effective yield on our investment strategy over the 5 years, we solve $421.24(1+i)^5 = 500$, which gives $i = 3.4875\%$.
- Even though it is true that the annuity we purchased from ABC yielded 6%, our overall investment strategy ultimately earned only 3.4875% since the payments received were not reinvested.

Reinvesting Annuity Payments

Assume that L is paid for an annuity that makes level payments of R at the end of each year for n years. Let i be the yield rate on the annuity. Then $L = R a_{\overline{n}|i}$. Suppose that as annuity payments are received, they are reinvested into Account X, which earns interest at an annual effective rate of j. Since R is deposited into Account X at the end of each year, the value of this account at the end of n years will be $R s_{\overline{n}|j}$.

This investment strategy required an initial investment of L, and yields an amount of $R s_{\overline{n}|j}$ at the end of n years. The effective rate of return on this strategy, k, can thus be found by solving $L(1+k)^n = R s_{\overline{n}|j}$.

Example 3.1 Kyle pays 3000 for a 10-year annuity immediate with annual payments of 400. As the payments are received, they are reinvested into an account that earns an annual effective rate j. Kyle's overall effective rate of interest earned is 4.8176%. Find j.

Reinvesting Interest Payments

- Assume that an amount of L is loaned for a period of n years at an annual effective rate i.
- Suppose that interest payments of Li are made at the end of each year, and the original amount of L is paid back at the end of year n.
- If the interest payments are reinvested into an account earning a rate of j, then the balance of this account at the end of n years will be $Lis_{\overline{n}|j}$.
- At time n, the lender will receive the principal repayment of L, plus the AV of the reinvestment account, which is $Lis_{\overline{n}|j}$.
- The lender's overall yield k can be found by solving $L(1+k)^n = Lis_{\overline{n}|j} + L$.

Example 3.2 Kara loans David 2000 at an annual effective rate of i. David makes annual interest payments at the end of each year, and repays the 2000 at the end of year 8. Kara reinvests the interest payments received from David into an account that earns 3% annual effective. Kara's overall rate of return over the course of the 8 years was 6.236%. Find i.

Example 3.3 Peter loans Harry 2400 at 8% annual effective. At the end of each year, Harry repays the interest accumulated over the course of the previous year, plus an additional 200. The loan is repaid after 12 such payments. As Peter receives payments from Harry, he reinvests them into Fund X, which earns interest at an annual effective rate of 5%. Find the rate of return earned by Peter over the course of the 12 years.

Multiple Reinvestment Accounts

You will occasionally encounter problems with multiple reinvestment funds. Dealing with such problems requires a combination of the methods discussed above.

Example 3.4 An investor purchases a bond for 1000. The bond will make “coupon payments” of 100 at the end of each year for 12 years, and a principal repayment of 1000 at the end of the twelfth year.

The coupon payments are reinvested into Fund X, which earns interest at an annual effective rate of 5%. At the end of each year, the accumulated interest from Fund X is deposited into Fund Y, which earns 3% annual effective.

Determine the investors rate of return over the 12 year period.

Example 3.5 Jesse pays P for an annuity that makes payments of 160 at the beginning of each year for 20 years. The annuity payments are reinvested into Fund X, which earns 10% annual effective. The interest earned by Fund X each year is withdrawn and deposited into Fund Y, which earns 6% annual effective. Jesse's annual yield rate over the 20 year period is 8%. Find P.

3.2 AMORTIZING A LOAN

Amortization is the process of settling a debt. Depending on the terms set when a loan is made, there are many different methods that can be used to repay the debt. The following example compares three possible methods.

Example 3.6

Pam, Cheryl, and Ray each borrow 2000 for 10 years at an annual effective rate of 6%. Pam makes no payments until the end of the 10 years, at which point she repays the loan with one lump sum payment. Cheryl pays the accumulated interest at the end of each year and repays the principal at the end of the 10 years. Ray repays the loan by making 10 level annual payments at the end of each year. Calculate the amount of interest paid by each of the three individuals.

Amortization Using A Level Annuity

Throughout the rest of this section, we will consider only cases in which debt is amortized using level annuities. Before looking at examples, we need to establish some notation and terminology. For simplicity, we will assume here that the payments occur on an annual basis. In general, the payment periods in am amortization problem could be quarters, months, weeks, or any other period of time.

- Let L represent the original **loan amount**. This is also called the **initial principal**.
- Let R be the level annual **payment**. Then $L = R\,a_{\overline{n}|}$ and $R = L \,/\, a_{\overline{n}|}$.
- Let B_t be the amount owed at time t. This quantity is referred to as the **unpaid balance** or the **outstanding principal** at time t. Note that $B_0 = L$.

Calculating Unpaid Balance

We will make frequent use of two different algebraic methods for calculating the unpaid balance of a loan at time t. These methods are called the *retrospective method* and the *prospective method*.

- **Retrospective Method.** Assume that no payments have been made against the debt. Then the amount owed at time t would be $L(1+i)^t$. If annual payments of R are made, however, then the outstanding balance would be reduced by the accumulated value of these payments. That is: $B_t = L(1+i)^t - R\,s_{\overline{t}|}$
- **Prospective Method.** Regardless of the number of payments that have been made up until this point, and regardless of the original amount of the loan, the currently outstanding balance must be the present value of all future payments that have yet to be made. Thus: $B_t = R\,a_{\overline{n-t}|}$.

Example 3.7

Doug borrows 50,000 at 6% convertible monthly. According to the original terms of the loan, the debt is to be repaid with level payments at the end of each month for 20 years, with no option for early repayment. At the end of 8 years, Doug renegotiates the terms of the loan. Under the new terms, he will pay the remaining balance with monthly payments lasting 6 more years, but his debt will now accumulate interest at 6.3% convertible monthly. Calculate the total amount of money that Doug saved by renegotiating the debt.

Calculator Tip: In the previous problem, the unpaid balance B_{96} can be calculated using the BA II as follows:

- [2ND] [CLR TVM] 240 [N] 0.5 [I/Y] 50000 [PV] [CPT] [PMT] 96 [N] [CPT] [FV]

Example 3.8

Cedric takes out a loan that is charged interest at an annual effective rate of 4%. He agrees to pay the loan back over the course of 30 years by making level payments at the end of each year. After 12 years, Cedric refinances his loan to obtain a lower rate. Under the terms of the refinance, he makes an immediate payment of 20,000 which is applied to the loan balance. His rate on the remaining balance is then lowered to 3%. Under the new terms of the loan, Cedric's annual payments for the remaining 18 years are 8127.08. Find the original loan amount.

Calculator Tip: The previous problem can be calculated using the BA II as follows:

- [2ND] [CLR TVM] 18 [N] 3 [I/Y] 8127.08 [+/-] [PMT] [CPT] [PV] [+] 20000 [=] [PV] 4 [I/Y] [CPT] [PMT] 30 [N] [CPT] [PV]

Amortization Tables

When amortizing a loan, each payment can be split into two pieces: the interest payment I_t and the principal reduction P_t. The interest portion is equal to the interest that has been accumulated since the last payment (i.e. $I_t = i \cdot B_{t-1}$). The principal portion of the payment is the amount by which the unpaid balance is reduced after the interest is paid (i.e. $P_t = R - I_t$).

An amortization table is a table that displays the values R, I_t, P_t, and B_t for each payment. Consider a loan with the following parameters: $L = 1000$, $R = 250$, $n = 5$, $i = 7.9308\%$. The amortization table for this loan is provided on the left below. The table to the right is the amortization table for an arbitrary loan with $n = 5$. We will use this general table to help us obtain formulas for directly calculating I_t and P_t without having to construct an amortization table.

t	R_t	I_t	P_t	B_t
0				1000
1	250	79.31	170.69	829.31
2	250	65.77	184.23	645.08
3	250	51.16	198.84	446.24
4	250	35.39	214.61	231.63
5	250	18.37	231.63	0
Totals	**1250**	**250**	**1000**	

t	R_t	I_t	P_t	B_t
0				$R\,a_{\overline{5}\rceil}$
1	R	$R(1-v^5)$	$R\,v^5$	$R\,a_{\overline{4}\rceil}$
2	R	$R(1-v^4)$	$R\,v^4$	$R\,a_{\overline{3}\rceil}$
3	R	$R(1-v^3)$	$R\,v^3$	$R\,a_{\overline{2}\rceil}$
4	R	$R(1-v^2)$	$R\,v^2$	$R\,a_{\overline{1}\rceil}$
5	R	$R(1-v^1)$	$R\,v^1$	0
Totals	$5R$	$5R-L$	L	

Interest and Principal Payments

We can use the table on the right above to make the following general observations about I_t and P_t:

- $I_t = R(1-v^{n+1-t})$ and $P_t = R\,v^{n+1-t}$.
- The values of P_t form a geometric sequence with common ratio $(1+i)$.
- $L = \Sigma P_t$.

Example 3.9

A loan of 5000 collects interest at an annual effective rate of 5% and is to be repaid with annual payments made over 12 years. Find the amount of interest paid and the principal repaid in the fifth installment.

Summary of Formulas

We summarize the formulas used for amortization in the table below.

Quantity	Definition	Formula 1	Formula 2
L	Original loan amount.	–	–
R	Level payment.	$R = L \,/\, a_{\overline{n}\rceil}$	–
B_t	Unpaid balance at time t.	$B_t = L(1+i)^t - R\,s_{\overline{t}\rceil}$	$B_t = R\,a_{\overline{n-t}\rceil}$
I_t	Interest portion of payment t.	$I_t = i \cdot B_{t-1}$	$I_t = R\left(1 - v^{n+1-t}\right)$
P_t	Principal portion of payment t.	$P_t = R - I_t$	$P_t = R\,v^{n+1-t}$

It is also important to note that the values of P_t form a geometric sequence with ratio $(1+i)$, and that $L = \Sigma\, P_t$.

Example 3.10 Gabe has a loan that is to be repaid with annual payments of 1000 at the end of each year for $2n$ years. The loan collects interest at an annual effective rate of 5.9%. The sum of the interest paid in year 1 plus the interest paid in year $n+1$ is equal to 1610. Find the amount of interest paid in year 8.

Example 3.11 Cora is repaying a loan by making payments of 2000 at the end of each quarter. The loan collects interest at a nominal rate of 8% convertible quarterly. The amount of interest paid in the tenth payment is 1271.51. Find the principle repaid with payment number 24.

Example 3.12 Bruce repays a loan by making payments at the end of each year for n years. The unpaid balance of the loan accumulates interest at a rate of 8% annual effective. The amount of interest paid in the final installment is 62.48. The total principal repaid at the time of the second-to-last payment is 6438.29. Calculate the principal repaid in the first payment.

Example 3.13 A loan is repaid over 15 years with level annual payments. The loan collects interest at 7% annual effective. The principal repaid with the fifth payment is 187. Find the loan amount.

Example 3.14 ABC Corp. borrowed 100,000 at a nominal rate of 6% convertible semiannually. The loan is to be repaid with level payments at the end of each six month period. The amount of interest paid in the eighth payment is 2516.82. Find the principle repaid with the fifteenth payment.

3.3 SINKING FUNDS

Assume that a borrower agrees to make interest payments on a loan at the end of each period. Since these interest payments do not repay any of the principal, the unpaid balance is once again equal to the original loan amount after each payment. Assume also that the borrower agrees to repay the original loan amount at some specified future time.

Suppose now that in addition to making the interest payments to the lender, the borrower also makes annual deposits into a side fund with the intent of eventually using the accumulated value of the side fund to repay the loan. A fund such as this side fund is called a **sinking fund**. Two reasons why the borrower might opt to repay a loan using the sinking fund method as opposed to the standard amortization method are: (1) the terms of the loan might not allow for payments (other than the last) to cover anything more than the interest, and (2) the sinking fund method is preferable to the amortization method if the sinking fund earns interest at a rate larger than what the original loan is being charged.

Before looking at examples, we will establish some notation and terminology relating to sinking funds.

- Let L be the original loan amount.
- Let i be the annual effective rate for the loan. Let j be the annual rate earned by the sinking fund.
- The size of the annual interest payments made to the lender are fixed at $I = i \cdot L$.
- The size of the annual sinking fund deposits are given by $SFD = L \,/\, s_{\overline{n}|j}$.
- The total amount paid by the borrower each year is $R = I + SFD$.
- The balance of the sinking fund at time t is given by $SFB_t = SFD\, s_{\overline{t}|j}$.

Example 3.15 A loan of 1000 is charged interest at 4% annual effective. The loan must be repaid in 10 years.

a) Assume that the loan is repaid using the standard amortization method by making level annual payments. Find the size of the payments.

b) Assume that the borrower makes interest payments to the lender at the end of each year and repays the original loan amount of 1000 at the end of year 10. The borrower also makes annual deposits into a sinking fund earning 6% annual effective in order to accumulate the 1000 to be repaid at time 10. Calculate the total amount paid each year by the borrower (including the interest payment and the sinking fund deposit).

c) Taking into account the effect of the sinking fund, determine the effective rate of interest that the borrower paid in the scenario outlined in Part (b).

Example 3.16 Julie borrows 20,000 for 18 years at an annual effective interest rate of i . She repays the loan using the sinking fund method. Her sinking fund earns an annual effective rate of 8%. Julie's total annual payment, including her interest payment and her sinking fund deposit, is equal to P . Had the effective rate on her loan been $2i$, then her total payment would have been $1.7P$. Find i .

Example 3.17 A loan of 60,000 is to be repaid over the course of 20 years. The borrower pays interest on the loan at the end of each year at a rate of 8%. The borrower also makes annual deposits into a sinking fund earning 6% with the intent of accumulating 60,000 in the sinking fund by the end of year 20. At the end of year 8, the rate earned by the sinking fund drops to 5%. Calculate the size of the sinking fund deposit for years 9 through 20.

Net Balance and Net Interest For the Sinking Fund Method

Under the sinking fund method, the size of the interest payments to the lender are level throughout the lifetime of the loan and the unpaid balance of the loan is the same after every payment. However, the balance of the sinking fund itself increases over time, as does the amount of interest earned by the sinking fund. This observation leads us to consider the concepts of net interest and net balance.

- The **net interest paid** at time t is denoted by I_t and is equal to the level interest payment minus the interest earned by the sinking fund. Thus, $I_t = I - j \cdot SFB_{t-1}$.
- The **net unpaid balance** at time t is denoted by B_t and is equal to the original loan amount minus the accumulated value of the sinking fund. Thus, $B_t = L - SFB_t$.

Example 3.18 A loan of 50,000 is repaid over 12 years by making annual interest payments at an effective rate of i, as well as level payments into a sinking fund earning 5% annual effective. The net interest paid during year 4 is 3400. Find the net interest paid during year 8.

Example 3.19 Trevor borrows 22,000 to be repaid in 10 years. He makes annual interest payments at 8% annual effective. Trevor also makes annual deposits of 1675 into a sinking fund earning an effective rate of i in order to accumulate 22,000 to repay the loan at the end of 10 years. Find Trevor's net balance at the end of year 7.

Example 3.20 Joanna will repay a loan over 12 years by making annual interest payments, as well as deposits of X into a sinking fund at the end of each year. The amount of interest earned by the sinking fund during the fourth year is $0.157625X$. The net amount of the loan immediately after the eighth payment is 2493.04. Find X.

3.4 VARYING PAYMENTS AND EQUAL PRINCIPAL REPAYMENT

Under the standard amortization method, as well as the sinking fund method, periodic payments are level. It is possible to establish a schedule for repaying a loan that utilizes varying payments instead. Consider that the payments could vary in any number of ways, it is not possible to establish general formulas to cover all such situations. When dealing with varying payments, one must apply general interest theory and annuity principles.

Example 3.21

A loan is to be repaid over five years with payments at the end of each month. The loan collects interest at a nominal rate of 6% convertible monthly. The first payment is 1000, and each later payment is 2% lower than the one preceding it. Find the unpaid balance at the end of year 3.

Example 3.22

Todd borrows X at an annual effective rate of i. The loan is to be repaid with payments at the end of each year for 14 years. The first payment is 700 and each subsequent payment decreases by 50. The amount of principal repaid in year 4 is equal to 325. Find X.

Example 3.23

A loan of 60,000 accumulates interest at an annual effective rate of 6%. The loan is to be repaid with payments at the end of each year for 32 years. The initial payment is X and each subsequent payment is X larger than the preceding payment. Find the amount of principal outstanding after payment number 19.

Equal Principal Repayment

One common method of utilizing varying payments to amortize a loan involves setting payments in such a way that an equal amount of principal is repaid with each installment. The details of this method are given below.

- Assume a loan of L accumulates interest at an annual effective rate of i.
- The loan is to be repaid with payments at the end of each year for n years.
- Denote the annual payment by R_t.
- We assume that each payment R_t repays an equal amount of principal given by $P_t = L / n$.
- Since the outstanding principal decreased by L / n with each payment, we have that $B_t = \frac{n-t}{n} L$.
- The amount of interest accumulated at the end of each year is thus given by $I_t = i \cdot B_{t-1} = \frac{n+1-t}{n} L \cdot i$.
- Combining some of the results from above, we see that $R_t = P_t + I_t = \frac{L}{n} + \frac{n+1-t}{n} L \cdot i$.
- Since L must be the PV of the loan payments, if follows that $L = \frac{L}{n} \cdot a_{\overline{n}|} + \frac{L\,i}{n} \cdot (Da)_{\overline{n}|}$.

Example 3.24

Ani borrowed 5000 from ABC Loans at 6% annual effective. The loan is to be repaid with payments at the end of each year for 20 years. Each payment will repay an equal amount of principal.

Immediately after the loan was made, ABC Loans sold the right to receive Ani's payments to XYZ investing for a price that will earn XYZ an annual effective return of 4%. Find the price paid by XYZ.

Example 3.25

Larry and David each borrow 4800.

Larry is charged an annual effective rate of 6%. He repays his debt by making level annual payments at the end of each year for 12 years.

David is charged an annual effective rate of i. He repays his debt by making payments at the end of each year for 12 years. David's payments each repay an equal amount of principal.

The total of Larry's payments is equal to the total of David's payments. Find i.

CHAPTER 4 – Bonds

4.1 BOND VALUATION

A bond is a mechanism for borrowing money that is often used by federal and local governments, as well as by large corporations. The purchaser of a bond (i.e. the lender) receives regular interest payments (called coupons) for a fixed period of years. On the maturity date of the bond, the lender receives a payment (called the redemption amount) that is generally equal to the original purchase price of the bond.

Before introducing the notation and formulas used for bond valuation, let's consider an introductory example.

Example 4.1

Garret pays 1000 to purchase a bond. The bond pays semiannual coupons (interest payments) at a nominal rate of 6% convertible semiannually and is redeemed for 1000 at the end of 10 years.

Two years later, immediately after receiving the fourth coupon, Garret decides to sell the bond to Stella. This gives Stella the right to collect the remaining coupon payments, as well as the redemption amount of 1000. Interest rates have dropped over the course of the two years, and the price that Garret charges Stella will allow her to earn a nominal semiannual rate of 4% on her investment.

a) What price did Stella pay for the bond?
b) What nominal semiannual rate did Garret actually earn during the two year period?

Bond Terminology and Notation
We will use the following notation and terminology when working with bonds.

- P is the current price of the bond.
- F is the face amount, or par value. This amount is used to determine the size of the coupons.
- C is the redemption amount. Unless otherwise stated, bonds are redeemable at par, meaning that $C=F$.
- n is the number of remaining coupon payments.
- r is the effective coupon rate per payment period. The size of the coupon is thus Fr .
- g is a special coupon rate that is occasionally used in formulas when $C\neq F$. It is defined by $Cg = Fr$.
- i is the effective interest rate per payment period earned by the purchaser of the bond.

We make the following comments regarding the terms introduced above:

- **Coupon Rate vs Interest Rate.** Bond problems involve two different rates. The coupon rate is used ONLY to determine the size of coupon payments. The interest rate is the rate that is actually used to price the bond once the coupons have been determined. The reason for the different rates is that interest rates can change over time, but the coupons are set when the bond is initially issued. See Example 4.1.
- **Price, Par Value, and Redemption Amount.** The price P is what is paid for the bond. The par value F is used only for determining the coupon size. The redemption amount C is what is actually repaid at redemption. Bonds we consider will generally be redeemable at par $(C=F)$. For a bond redeemable at par, if $r = i$ then the price is equal to the redemption amount and $P=C=F$.
- **Special Coupon Rate.** Assume that a bond is not redeemable at par. If we replace the coupon rate with the special coupon rate, then we can now consider the bond to be redeemable at par.

Bond Valuation Formulas

We will use the following two formulas to price bonds:

- Basic Bond Valuation Formula: $P = F\,r\,a_{\overline{n}|i} + C\,v^n$
- Premium/Discount Formula: $P = C + (F\,r - C\,i)a_{\overline{n}|i}$

The first formula can be derived using basic annuity concepts. The premium/discount formula can be derived from the basic formula using algebraic methods. The basic formula will be the one that we use for most bond problems. However, some problems are more easily solved using the premium discount formula. The advantage of the P/D formula is that the variable n only appears in one place in the formula.

Note on rates: It is important to note that the rates r and i used in the bond formulas above are effective rates for the stated coupon period. Most bonds pay semiannual coupons. In that case, r and i are both **effective semiannual** rates.

We now consider two basic examples utilizing the bond valuation formulas.

Example 4.2

A 15-year 1000 par value bond yields 4% convertible semiannually. Coupons are paid semiannually. The bond is redeemable at par.

a) Find the purchase price of the bond if it pays coupons at 3% convertible semiannually.
b) Find the purchase price of the bond if it pays coupons at 5% convertible semiannually.

Example 4.3

A 2000 par bond pays coupons semiannually at 5% per annum and is redeemable at par after 10 years. The price of the bond is 1900. Find the nominal semiannual yield rate of the bond.

A zero coupon bond is a bond that pays no coupons. The price of a zero coupon bond is simply the present value of its redemption amount. Example 4.4 involves such a bond.

Example 4.4

Hailey buys three bonds. Each bond has a par value of 1000, matures in n years, and is priced to yield an annual effective rate of i. You are given:

i) The first bond is a zero coupon bond and has a price of 402.78.
ii) The second bond pays 8% annual coupons and has a price of 1167.22.
iii) The third bond pays 5% annual coupons and has a price of P.

Find P.

Example 4.5

Elliot purchases a 20-year, 1000 par bond. The bond pays semiannual coupons at a rate of 6% convertible semiannually and is priced to yield an annual effective rate of i. As the coupon payments arrive, Elliot reinvests them into an account earning 5% convertible semiannually. At the end of the 20 year period, Elliot's overall effective annual yield is 7%. Calculate i.

Example 4.6

Darlene pays P for a 15 year, 2000 par bond paying semiannual coupons at 6%. After 6 years, immediately after the 12th coupon, Darlene sells the bond to Angela at a price that yields Darlene a rate of 8% convertible semiannually and yields Angela a rate of 7% convertible semiannually. Find P.

4.2 PREMIUM AND DISCOUNT

As we have seen in previous examples, if $r > i$ for a par-value bond, then $P > C$. Similar, if $r < i$ for a par-value bond, then $P < C$. We will now introduce terminology to refer to these two situations.

- **Premium.** If $P > C$ for a bond, then the bond is said to be purchased at a *premium*. For par-value bonds, this occurs when $r > i$. In general, a bond is at a premium if $g > i$.
- **Discount.** If $P < C$ for a bond, then the bond is said to be purchased at a *discount*. For par-value bonds, this occurs when $r < i$. In general, a bond is at a premium if $g < i$.

Amount of Premium or Discount

- For a bond purchased at a premium, the value $P - C$ is referred to as the **amount of premium**.
- For a bond purchased at a discount, the value $C - P$ is referred to as the **amount of discount**.

If using the P/D formula $P = C + (Fr - Ci)a_{\overline{n}|i}$ to price a bond, the quantity $(Fr - Ci)a_{\overline{n}|i}$ will be positive if the bond is sold at a premium and negative if the bond is sold at a discount. In either case, $|Fr - Ci|a_{\overline{n}|i}$ will be equal to the amount of premium or discount.

Example 4.7

A 15-year 1000 par value bond yields 4% convertible semiannually. Coupons are paid semiannually The bond is redeemable at par. Find the amount of premium or discount if:

a) The bond pays coupons at 3% convertible semiannually.
b) The bond it pays coupons at 5% convertible semiannually.

Book Value of a Bond

The book value of the bond is the current price of the bond, if it were to be resold at the same yield rate as when it was purchased. Alternately, one can think of the book value of a bond as being the current outstanding loan balance. We will use B_t to refer to the book value immediately after payment number t. In the next section, we will discuss the book value of a bond at times between two coupon payments. For now, however, we are only interested in B_t at times immediately after a coupon payment has been made. Notice the following:

- $B_0 = P$ since P is the amount that is initially borrowed by the bond issuer.
- $B_n = C$ since C is the amount that is ultimately repaid by the bond issuer.
- At any other time, B_t is equal to the PV of all future payments. It follows that $B_t = Fr\,a_{\overline{n-t}|i} + Cv^{n-t}$.

Noting that $B_0 = P$ and $B_n = C$, we can make the following observations about the book value of a bond:

- **Premium.** If $P > C$, then the book value decreases over time. This is because the coupon payments exceed the interest accumulated by the outstanding balance. The excess is applied to the principal.
- **Discount.** If $P < C$, then the book value increases over time. This is because the coupon payments are smaller than the interest accumulated by the outstanding balance. The deficit is added to the principal.
- **Neither.** If $P = C$, then $B_t = P = C$ after every coupon payment. In this case, the coupon payments are exactly equal to the interest accumulated on the loan.

We illustrate these concepts with the following example.

Example 4.8

Construct amortization tables for bonds with the following parameters:

a) $C = 1000$, $n = 3$, $r = 10\%$, $i = 8.0578\%$, $P = 1050$
b) $C = 1000$, $n = 3$, $r = 10\%$, $i = 12.0848\%$, $P = 950$
c) $C = 1000$, $n = 3$, $r = 10\%$, $i = 10\%$, $P = 1000$

Write-Down of Premium and Write-Up of Discount

Let I_t be the interest accumulated on the outstanding balance of the bond at time t. Let $P_t = \left|I_t - Fr\right|$. Then P_t is the absolute value of the difference between the accumulated interest and the coupon payment.

- For a bond purchased at a **premium**:
  - P_t is subtracted from the book value. That is, $B_t = B_{t-1} - P_t$.
  - P_t is referred to as the **write-down of premium**, or the **amount for amortization of premium**.
  - It can be shown that $P_t = (Fr - Ci)v^{n-t+1}$.
- For a bond purchased at a **discount**:
  - P_t is added to the outstanding balance. That is, $B_t = B_{t-1} + P_t$.
  - P_t is referred to as the **write-up of discount**, or the **amount for accumulation of discount**.
  - It can be shown that $P_t = (Ci - Fr)v^{n-t+1}$.
- In either case, we have that $P_t = \left|Fr - Ci\right|v^{n-t+1}$.
- It is important to note that the values P_t form a geometric sequence with common ratio $(1+i)$.

Example 4.9

A 15 year, 5000 par value bond pays semiannual coupons at 8% and is purchased to yield 6% convertible semiannually.

a) Find P_{10}, the write-down of premium in the 10th coupon payment.
b) Find I_{10}, the interest portion of the 10th coupon payment.
c) Find B_{10}, the book value after the 10th coupon payment.

Premium/Discount Formula

As the name implies, the P/D bond valuation formula can be useful in problems involving premium or discount.

Example 4.10

A 1000 par, 12-year bond pays semiannual coupons. The bond is purchased at a discount to yield 8% compounded semiannually. The amount for accumulation of discount in the 18th coupon is 15. Find the amount of discount in the original purchase price.

Example 4.11

A 1000 par, 12-year bond pays 5.5% annual coupons and is purchased at a discount to yield 8.5% annually. The write-up in value during the first year is 8.48. Find the purchase price of the bond.

Comparing Book Values

Thinking of book value for a bond as the outstanding balance of the loan allows us to construct a relationship between the book values of a bond at two different times. Consider two times t_1 and t_2 such that $t_1 < t_2$. Let $k = t_2 - t_1$. Then $B_{t_1} = Fr\,a_{\overline{k|}i} + B_{t_2}v^k$.

Example 4.12

Piper purchases an n-year 1000 par bond. The bond pays annual coupons at a rate of 6%. The book value of the bond at the end of 3 years is 1161.09. The book value of the bond at the end of 5 years is 1145.24. Find the price of the bond.

4.3 PRICES BETWEEN COUPON DATES

Bonds are regularly bought and sold on the market. It is certainly not always true that the time of purchase for a bond will fall immediately after a coupon has been paid. In this section, we will consider two methods for pricing bonds at times that fall between two coupon payments.

Full Price

The *full price* (also called the *dirty price*, *flat price*, or *price-plus-accrued*) is equal to the book value of the bond immediately after the most recent coupon, accumulated at interest. Thus, the full price of a bond k coupon periods (with $0 < k < 1$) after the *n*th coupon is given by the formula $B_{t+k} = (1+i)^k B_t$.

Market Price

The *market price* (also called *clean price*, or simply *price*) is the full price minus the portion of the next coupon that has so far been "accumulated". For instance, at time $t + k$, the book value is B_{t+k} and the amount of the coupon that has been "accumulated" would be $k\,F\,r$. This, the market price at this time is $B_{t+k} - k\,F\,r$.

Full Price vs. Market Price

It is important to note that the full price is the actual price of the bond at time $t + k$. If a bond is sold at time $t + k$, then B_{t+k} is the price paid for the bond. The market price is essentially an estimation. To understand the purpose of considering the market price, notice that the full price does not change continuously over time. Assume that a coupon payment has just been made. The dirty price will continuously increase as interest is accumulated over the course of the next coupon period. However, once the coupon payment has been made, the dirty price will drop by the amount of the coupon, causing a discontinuity in the price. The market price provides a continuous (although technically less correct) estimation of the bond price. Some financial institutions report bond prices using the full price and some use the market price.

Summary of Formulas

- Full Price: $B_{t+k} = (1+i)^k B_t$
- Market Price: $B_{t+k} - k\,F\,r$

Example 4.13

A 5-year, 1000 par bond pays semiannual coupons at a rate of 10%. The bond is purchased to yield 7% compounded semi-annually.

a) Find the dirty price of the bond 15 months after its purchase, assuming the same yield.
b) Find the clean price of the bond 15 months after its purchase, assuming the same yield.

Example 4.14

A 2000 par value 12 year bond pays 6% semiannual coupons. The yield rate is 8% convertible semiannually.

a) Find the dirty price of the bond 7.2 years after its purchase, assuming the same yield.
b) Find the clean price of the bond 7.2 years after its purchase, assuming the same yield.

4.4 CALLABLE BONDS

For the types of bonds we have been discussing, the maturity date is set when the bond is created. That is to say that the date on which the borrower repays the redemption amount is determined from the outset and the borrower is not given the option of early repayment. A **callable bond** is one in which the borrower is given the option to repay the redemption amount of the bond prior to the originally stated maturity date. Typically, a callable bond will have a specific range of dates leading up to the maturity date during which early repayment is an option. Some callable bonds will also have a variable redemption amount that depends on the date on which the bond is called.

A crucial thing to understand about callable bonds is that the stated yield rate assumes that the bond will be held until maturity. If the bond is called early, that will almost certainly have an effect on the yield rate realized by the lender. Whether the yield increases or decreases depends on whether the bond was sold at a premium or discount. This concept is illustrated in the following two examples.

Example 4.15

A 10-year 1000 par bond delivers 10% annual coupons. The bond was purchased for 1050 and can be called at any point after 5 years.

a) Determine the yield on the bond if it is not called early.
b) Determine the yield if the bond is called immediately after the 8th coupon is paid.

Example 4.16

A 10-year 1000 par bond delivers 10% annual coupons. The bond was purchased for 950 and can be called at any point after 5 years.

a) Determine the yield on the bond if it is not called early.
b) Determine the yield if the bond is called immediately after the 8th coupon is paid.

Effects of Early Redemption

- For a bond sold at a premium, earlier redemption dates will result in a lower yield. This can be remembered using the mnemonic device PEW, which stands for: “Premium: Earlier is Worse”.
- For a bond sold at a discount, earlier redemption dates will result in a higher yield. This can be remembered using the mnemonic device DEB, which stands for: “Discount: Earlier is Better”.

Example 4.17

Peter pays 1514.52 for a 24-year par value bond paying coupons semiannually at a rate of 6%. The bond can be called at par on any coupon date starting at the end of year 17. The price paid by Peter guarantees him a yield of at least 5% compounded semiannually.

a) Calculate the par value of this bond.
b) Calculate the highest yield that Peter might earn on this bond.

Example 4.18

Bruce purchases a 16-year 10,000 par bond paying 5% semiannual coupons. The bond is callable at par on any coupon date beginning at the end of year 9. The price paid by price guarantees him a yield of at least 7% convertible semiannually.

a) Find the price paid by Bruce.
b) Calculate the highest yield that Bruce might earn on this bond

4.5 SPOT RATES AND FORWARD RATES

Up to this point, when calculating present values we have generally been provided with one rate of interest that we would use to calculate the present value of any payment, regardless of when it occurred. In practice, however, the yield rate that you can get on an investment such as a bond tends to depend on the length of time until the bond matures. Longer term bonds usually (but not always) have higher yields than bonds with shorter terms. Two reasons why a bond purchaser might demand a higher return for a long-term bond are given below.

1. Assume a company issues 5-year bonds and 30-year bonds. There is a greater risk of the company defaulting on the 30-year bond than the 5-year bond. An investor in the 30-year bond would likely want a higher return to compensate for the additional risk.
2. An individual investing in a long-term bond will have their money tied up in the bond for an extended period of time, and will be forgoing the ability to invest their money in other opportunities that might come along later.

Spot Rates

A **spot rate** is a yield rate for a zero-coupon bond. More specifically, the n-year spot rate, denoted by s_n, is the annual effective yield for n-year zero-coupon bonds currently on the market. Thus, the price today for an n-year zero-coupon bond paying 1 can be calculated using $P = 1/\left(1+s_n\right)^n$. A table or graph that reports spot rates for a range of years is called a **yield curve**.

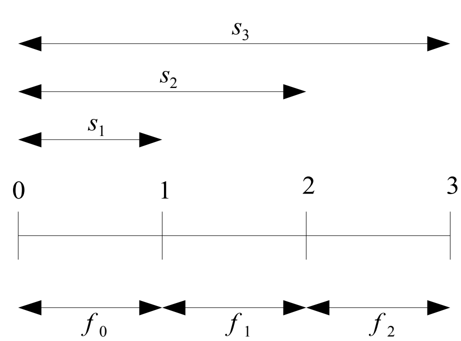
Example 4.19 Consider the yield curve provided below.

Years until Maturity	1	2	3	4
Yield on a Zero-Coupon Bond	5.00%	6.00%	6.75%	7.25%

a) Find the prices of zero-coupon bonds maturing for 100 in 1, 2, 3, and 4 years.
b) Find the price of a four-year 1000-par bond paying annual coupons of 50.
c) Find the yield rate for the bond whose price was calculated in Part b.

Forward Rates

Recall that a spot rate is an effective rate for a multiple year period beginning today. That said, any yield curve stated in terms of spot rates will imply specific annual effective rates of interest for any one year period covered by the yield curve. Such an effective annual rate is called a **forward rate**. The forward rate f_n is the effective rate for year $n+1$, as implied by the given yield curve. The diagram to the right illustrates this concept. The relationships between spot and forward rates are provided below.



- $1+f_n = \dfrac{\left(1+s_{n+1}\right)^{n+1}}{\left(1+s_n\right)^n}$
- $\left(1+s_n\right)^n = \left(1+f_0\right)\cdot\left(1+f_1\right)\cdot\ldots\cdot\left(1+f_{n-1}\right)$

Example 4.20 Find the forward rates implied by the yield curve given below.

Years until Maturity	1	2	3	4
Yield on a Zero-Coupon Bond	5.00%	6.00%	6.75%	7.25%

Locking In Forward Rates

Interest rates change frequently. Knowing a yield curve today doesn't tell you what rates will be in the future. When we say that the forward rate f_n is the effective rate for year $n+1$, we don't mean that this is what the one-year rate *will* be when year $n+1$ arrives. We mean that, according to current spot rates, this is the rate that should be used for year $n+1$ when doing calculations today. That said, it is possible to set up transactions that will guarantee you an effective rate of f_n during year $n+1$. The steps in doing so are described below.

1. Borrow 1, agreeing to repay $(1+s_n)^n$ in n years.
2. Immediately reinvest the 1 that was borrowed into an ($n+1$)-year bond paying $(1+s_{n+1})^{n+1}$.
3. There is no net investment at time 0.
4. A liability of $(1+s_n)^n$ will occur at time n, and an asset of $(1+s_{n+1})^{n+1}$ will be paid at time $n+1$.
5. The effective yield during year $n+1$ is given by $\dfrac{(1+s_{n+1})^{n+1}}{(1+s_n)^n} - 1 = f_n$.

Example 4.21 Current prices for 1000 par zero-coupon bonds are given below.

Years until Maturity	1	2	3
Price	952.38	X	843.64

The one-year forward rate for year 2 is 6%. Find X.

Example 4.22 The current five-year spot rate is 8%. The forward rate for year 2 is 6%. The current spot rate for a three-year bond purchased at time 2 is 10%. Find the one-year spot rate.

Example 4.23 Current spot rates for n-year zero coupon bonds are provided below.

Years until Maturity	1	2	3	4
Yield on a Zero-Coupon Bond	3.00%	4.00%	4.50%	5.00%

A five-year 1000 par bond pays annual coupons of 6% and is priced according to current spot rates. The bond price results in an annual effective yield of 6.5%. Find the five-year spot rate.

CHAPTER 5 – Yield Rates

5.1 DETERMINANTS OF INTERESTS RATES

Supply and Demand of Money

Consumption refers to the expenditure of money to purchase goods or services. **Interest** can be viewed as either the compensation for delaying consumption or the cost for advancing consumption, depending on whether it is viewed from the perspective of the lender or the borrower.

We begin by discussing the meaning of interest when viewed from the differing perspectives of the lender and the borrower.

- **Lender Perspective:** From the lender's perspective, interest is the compensation received for delaying consumption. As interest rates increase, it becomes more enticing to delay consumption, and more individuals with money are willing to lend that money. The availability of money to borrow will increase.
- **Borrower Perspective:** From the borrower's perspective, interest is the cost associated with making a purchase when the money for that purchase is not currently available. Thus, it can be viewed as the cost for advancing consumption. As interest rates increases, it becomes easier to

Effects of Interest Rates on Borrowing and Lending

As interest rates increase, it becomes more enticing to delay consumption by lending money. As a result, the availability of money to borrow will increase with the interest rate. On the other hand, larger interest rates represent higher costs for borrowing money. So, as rates increase, the number of people willing to borrow will decrease. If rates are very low, then there will be many people interested in borrowing, but few people willing to lend. When rates are very high, there will be many people willing to lend, but few people willing to borrow the money that is available.

It stands to reason that within a particular lending market, there will be a certain interest rate at which the supply of money to borrow will be equal to demand for borrowing. Economic theory suggests that the current interest rate within that market should tend toward this **equilibrium rate** at which the supply and demand for money are equal. There are many factors that effect interest rate levels, but this supply and demand perspective provides us with a simplified view of how interest rates are determined within a lending market.

Components of Interest Rates

In this section, we will see how interest rates can be decomposed into several different component rates, which are called **interest rate determinants**. Throughout this section, we will represent interest rates, and their determinants, as continuously compounded rates. The reason for this is that the relationship between an interest rate and its determinants is more easily expressed in terms of continuously compounded rates than with annual effective rates.

We will discuss four interest rate determinants: the **real risk-free rate**, the **maturity risk**, the **default risk premium**, and the **inflation premium**. The current interest rate, which we will denote as R in this section, will be considered to be the sum of these four components.

Real Risk-Free Rate

The **real risk-free rate**, denoted by *r*, represents the true increase in purchasing power that the lender would expect to see as the result of the loan, in the absence of risk factors such as maturity risk, default risk, and inflation (all of which will be discussed later). The real-risk free rate acts as a "base rate" used when calculating the true interest rate, *R*.

Maturity Risk

As a general principle, lenders usually demand higher rates when making long term loans than they do when making short term loans. We will discuss the reasons for this momentarily, but the basic idea is that longer term loans carry a higher level of uncertainty for the lender, and so lenders insist on receiving a higher rate to compensate them for this additional risk. We will use r_M to denote the rate determinant associated with maturity risk, also called the **maturity risk premium.** When maturity risk is the only type of risk being considered by the lender, the interest rate is given by $R = r + r_M$.

Example 5.1

Suppose the three year yield curve is given by the forward rates $f_0 = 4\%$, $f_1 = 5\%$, and $f_2 = 7\%$, expressed as continuously compounded rates of interest. Assume that the real risk-free rate *r* for short-term loans is equal to f_0 . Determine the maturity risk premium for a three year loan.

There are several theories related to the general principle that lenders require higher compensation for making long-term loans. Four of the most important theories are discussed below.

- **Market Segmentation Theory.** The market segmentation theory assumes that individual lenders and borrowers typically enter the market with a preferred loan term already in mind. As a result, the lending market naturally segments itself based upon the loan terms desired by the individuals within the market. For simplicity, assume that loans are only available in 5, 10, and 20 year terms. Then the market segmentation theory predicts that there will be three distinct markets: One for 5-year loans, one for 10-year loans, and one for 20-year loans. Each of these markets will have its own supply and demand curves, and could thus each have its own distinct interest rates. The market segmentation theory allows for the possibility for rates to be different for loans of different terms, but does not predict whether long-term rates will be higher or lower than short-term rates.

- **Liquidity Preference Theory.** When a lender makes a loan, they relinquish access to those funds during the term of the loan. This represents a loss of opportunity, since the lender will not be able to use those funds to take advantage of a better investment opportunity, should one come along prior to the maturity date of the original loan. The liquidity preference theory, also called the opportunity cost theory, asserts that lenders naturally prefer shorter-term loans to maintain flexibility in how they make their investments. As a result, lenders would thus demand a higher rate when committing their funds to a long-term loan.

- **Preferred Habitat Theory.** This theory builds onto the market segmentation theory by also asserting that individuals might be compelled to take a loan that is not of their preferred term, if the compensation for doing so was sufficiently high. For example, a borrower seeking a long-term loan, might be tempted to take a short-term loan instead, if the rate in the short-term market was sufficiently low.

- **Expectations Theory.** This theory states that long-term rates provide information about expected short-term rates in the future. For example, given the three-year spot rate s_3 and four-year spot rate s_4 , one can calculate the expected one year forward rate f_3 inferred by these spot rates.

Default Risk Premium

When a borrower fails to repay a loan, they are said to **default** on the loan. Every loan carries with it some risk of default. Lenders will generally attempt to assess the magnitude of this risk, and adjust the interest rate in order to compensate for the risk that the borrower will default. The **default risk premium**, denoted by s, is the continuously compounded risk premium charged by the lender in order to offset the default risk for the loan. If default risk is the only type of risk being accounted for in the interest rate, then we have $R = r + s$.

The size of the risk premium s depends on whether or not the lender can expect to recover any part of the loan amount in the case of a default. The next example considers an example where no money is recovered when the borrower defaults.

Example 5.2

A lending organization groups its borrowers into three risk categories: low-risk, medium risk, and high-risk. Based on past information, the lender expects that for five-year loans, 2% of all low-risk clients will default , 5% of all medium-risk clients will default, and 10% of all high-risk clients will default. The lender does not expect to receive a partial payment when a borrower defaults.

The lender would like to achieve an expected continuously compounded return of $r = 4\%$ on five-year loans made to borrowers in each risk group. Determine the rate that the lender should charge to each group, as well as the default risk premium for each group.

In the next example, we will calculate the default risk premium under the assumption that the lender is able to recover a portion of the loan amount in the case of a default.

Example 5.3

A lender is making a three-year loan to a borrower. Based on the borrower's financial history, the lender assesses that there is an 8% chance that the borrower will default on the loan. The lender collects collateral equal to 20% of the repayment amount of the loan. The lender will claim this collateral in the case that the borrower defaults. Assuming that the lender would like to see an expected return of 6% on the loan, determine the true rate that should be charged, as well as the default risk premium.

Inflation

Prices of goods and services change over time, with a tendency to increase. This effect is know as **inflation**. The rate of increase is called the **inflation rate**. In the United States, the inflation rate is typically measured by one of two indexes, the **Consumer Price Index (CPI)** or the **Producer Price Index (PPI)**. The details on how these indexes are calculated differ, but the idea is similar. Each index tracks the price of a specific goods and services over time, and calculates a weighted average of current prices to determine the current value of the index. The rate of change in either of these indexes serves as an estimate for the inflation rate.

Example 5.4

The value of the CPI two years ago today was 237.42. The value of the index today is 246.52. Use these values to estimate the continuously compounded rate of inflation over the last two years.

A consequence of inflation is that the inherent value, or purchasing power, of a single unit of currency tends to decrease over time as prices increase. When a lender makes a loan, they need to account for inflation when setting their desired interest rate. If a lender sets an interest rate of 2%, but prices increase by a rate of 3% during the term of the loan, the the amount received by the lender at maturity will be numerically greater than the original loan amount, but will have a smaller amount of purchasing power.

The inflation rate is never known in advance. If it were, then the lender could simply account for inflation by adding the inflation rate to the desired real risk-free rate for the loan. The following example illustrates this idea.

Example 5.5

Consider a four-year loan of 1000. Suppose that there is no risk of default for the loan.

a) Assuming that there is no inflation, the lender requires a return of 6%. Calculate the repayment amount required by the lender.

b) Assume that the continuously compounded inflation rate for the next four years is known to be 1.5%. Calculate the amount of money that would carry the same amount of purchasing power as the repayment amount calculated in Part (a).

c) To account for the effects of inflation, the lender demands an interest rate that would yield a repayment amount equal to the amount found in Part (b). Determine the interest rate charges on the loan.

As mentioned above, it is unrealistic to assume that the interest rate is known in advance. One way in which a lender can account for the effects of inflation is to include **inflation protection** in the loan. In an inflation-protected loan, a desired interest rate will be set, from which a base repayment amount can be calculated. The actual amount repaid by the borrower when the loan matures will be this base amount adjusted according to the actual inflation observed during the term of the loan. In other words, the final repayment amount will be set so as to yield a rate of $R = r + i_a$, where r is the desired real risk-free rate, and i_a is the actual observed rate of inflation during the loan term. Note that this does not account for default risk. To additionally account for default risk, one would need to add in a risk premium, yielding $R = r + s + i_a$.

Example 5.6

Consider a 6-year loan of 1000 with inflation-protection. The loan agreement specifies a continuously compounded interest rate of 4%, with an inflation adjustment determined by the percentage increase in the CPI during the term of the loan. Assume that the CPI is equal to 232.10 when the loan is entered into, and is equal to 258.57 when the loan is repaid. Determine the amount repaid by the borrower.

While insurance protection provides a good solution for the lender, it might not be desirable to the buyer, who would likely prefer to know the exact amount that they would eventually be required to repay when entering into the loan. If inflation protect is not an option, then the lender might simply add on an estimate of what they expect the inflation rate to be based on recent history. Denote this estimate by i. If we assume that the rate includes premiums for inflation and default risk, then the interest rate would be given by $R = r + s + i$

Example 5.7

Consider a four-year loan of 1000. The lender desires a real risk-free rate of 5%. The lender estimates that there is a 10% chance of the borrower default. If the borrower does default, the lender anticipates that he will be able to recover 25% of the repayment value of the loan. The lender estimates that the continuously compounded rate of inflation will be 1.75% over the next four years. Determine the interest rate that the lender should charge.

5.2 DISCOUNTED CASH FLOW ANALYSIS

Net Present Value

- Assume a company is considering a project that is expected to require several investments, but is also expected to generate several payments.
- We will call the investments required *cash outflows*, or *liabilities.* If an outflow occurs at time *t*, we will denote it by L_t .
- The income generated for the company will be called *cashed inflows*, or *assets*. If an inflow occurs at time *t*, we will denote it by A_t .
- We generically refer to the collection of cash inflows and cash outflows as *cash flows*. If a cash flow occurs at time *t*, we often denote it by CF_t . To distinguish between liabilities and assets when using this notation, liabilities are set to be negative.
- Let PV_L be the present value of all liabilities associated with the project and let PV_A be the present value of all assets.
- The *net present value (NPV)* of the project is given by $NPV = PV_A - PV_L$.
- If $NPV > 0$, then the project is a good venture of the company. If $NPV < 0$, then the project will generate net losses for the company.
- When calculating NPV, an interest rate must by chosen. The rate used is called the *cost of capital* or the *interest preference rate*. It is generally the interest rate at which a company is able to borrow and lend money.

Example 5.8

Consider the following two cash streams:

i) $A_0 = 720,\ \ L_1 = 1700,\ \ A_2 = 100$

ii) $A_0 = 235,\ \ L_2 = 250$

Compare the NPV of these two cash streams using $i = 4\%$ as well as $i = 8\%$.

Internal Rate of Return

- Given a series of cash flows, the *internal rate of return (IRR)* of the cash flows is the interest rate at which the NPV of the cash flows is zero.
- The IRR of a series of cash flows is not necessarily unique.
- It is often impractical to calculate the IRR without using a financial calculator or a computer.

Calculating NPV and IRR with the BA II Plus

The BA II Plus can be used to calculate NPV and IRR for an irregular series of cash flows. The process can be somewhat complicated, however. A few examples are provided below.

1. Assuming that $i = 4\%$, show that the NPV of the following series of cash flows is 116.9698:
   $CF_0 = -400,\ CF_1 = 0,\ CF_2 = 200,\ CF_3 = 0,\ CF_4 = 100,\ CF_5 = 300$.
   - [CF] [2ND] [CE/C] 400 [+/-] [ENTER] [↓] 0 [ENTER] [↓] [↓] 200 [ENTER] [↓] [↓] 0 [ENTER] [↓] [↓] 100 [ENTER] [↓] [↓] 300 [ENTER] [↓] [↓] [NPV] 4 [ENTER] [↓] [CPT]

2. Change the interest rate in the previous problem to $i = 10\%$ to get a NPV of 19.8670.
   - [↑] 10 [ENTER] [↓] [CPT]

3. Assuming that $i = 6\%$, show that the NPV of the following series of cash flows is 113.3178:
   $CF_0 = 0,\ CF_1 = -200,\ CF_2 = 100,\ CF_3 = 100,\ CF_4 = 100,\ CF_5 = -300,\ CF_6 = 200,\ CF_7 = 200$.
   - [CF] [2ND] [CE/C] 0 [ENTER] [↓] 200 [+/-] [ENTER] [↓] [↓] 100 [ENTER] [↓] 3 [ENTER] [↓] 300 [+/-] [ENTER] [↓] [↓] 200 [ENTER] [↓] 2 [ENTER] [NPV] 6 [ENTER] [↓] [CPT]

4. Show that the IRR for the following series of cash flows is 7.9388%:
   $CF_0 = -600,\ CF_1 = 0,\ CF_2 = 0,\ CF_3 = 0,\ CF_4 = 200,\ CF_5 = 200,\ CF_6 = 500$.
   - [CF] [2ND] [CE/C] 600 [+/-] [ENTER] [↓] 0 [ENTER] [↓] 3 [ENTER] [↓] 200 [ENTER] [↓] 2 [ENTER] [↓] 500 [ENTER] [↓] [↓] [IRR] [CPT]

5.3 DOLLAR-WEIGHTED AND TIME-WEIGHTED RETURNS

Suppose you have an investment portfolio whose rate of return varies over time. Suppose also that you make occasional withdrawals from, or deposits into the portfolio. In this section we will consider two different methods of measuring the performance of such a portfolio: The dollar-weighted return, and the time-weighted return.

Dollar-Weighted Return (DWR)

As discussed in Section 5.1, the IRR is the interest rate at which the net present value of all of the transactions is 0. This could be restated by saying that the NPV of the deposits is equal to the NPV of the withdrawals. These NPV expressions can be large-degree polynomials, and thus calculating the IRR can be impractical without a computer or financial calculator. The dollar-weighted return (DWR) is a simple interest approximation of the IRR. The equations required to solve the DWR are linear in i, and thus much easier to solve.

The IRR (and hence DWR) can be heavily affected by the transactions made for the portfolio. For instance, assume that over the course of a year, a specific portfolio has a significant "up" period followed by a significant "down" period. Suppose that two investors both invest in the same portfolio, but one investor is invested only during the up period, the the other investor is only invested during the down period. The two investors will have drastically different DWRs, even though they were invested in the same portfolio.

If we wish to measure the performance of a portfolio without considering the effects of inflows and outflows, we can use the time-weighted return.

Time-Weighted Return (TWR)

Assume we wish to measure the performance of the portfolio on its own merits, without regards to any transactions posted to the account. In this case we can use the time-weighted return (TWR). The TWR is calculated by spitting the time period of concern into intervals of constant return, and then calculating the effective return from the returns of the smaller sub-periods. Although TWR doesn't take into account the effect of any transactions to the account, these transactions often have to be considered when calculating the returns during the shorter time periods.

Example 5.9

A fund collects interest at a nominal rate of 20% convertible semi-annually for $0 \le t \le 0.5$ and at a nominal rate of 10% convertible semi-annually for $0.5 \le t \le 1$.

a) Find the TWR of this fund.
b) Assume 100 is deposited into the account at $t = 0$. Find the DWR.
c) Assume that 100 is deposited at $t=0$ and 10 is deposited at $t=1/2$. Find the DWR.
d) Assume that 10 is deposited at $t=0$ and 100 is deposited at $t=1/2$. Find the DWR.

Calculating DWR and TWR from Transaction Information

Assume we are given (or have constructed) a table such as the following one containing transaction information for a fund. We will now discuss how to use such a table to calculate the DWR and TWR.

Time	$t = t_1 = 0$	$t = t_2$	$t = t_3$	$t = t_4$	$t = t_5 = 1$
Beginning Balance	0	B_2	B_3	B_4	B_5
Transaction	T_1	T_2	T_3	T_4	T_5
Ending Balance	E_1	E_2	E_3	E_4	0

Finding DWR

- The DWR is calculated by using the middle row containing the transaction information.
- Let $\Delta_i = 1 - t_i$ be the time elapsed between $t = t_i$ and $t = 1$.
- The fund is assumed to begin and end with a 0 balance, so we can set up the following equation for DWR: $T_1(1 + i\Delta_1) + T_2(1 + i\Delta_2) + T_3(1 + i\Delta_3) + T_4(1 + i\Delta_4) + T_5 = 0$.
- Solving this equation for i yields: $DWR = \dfrac{-(T_1 + T_2 + T_3 + T_4 + T_5)}{T_1\Delta_1 + T_2\Delta_2 + T_3\Delta_3 + T_4\Delta_4}$.

Finding TWR

- Notice that the accumulation factor for the time period $[t_i,\ t_{i+1}]$ is given by $\dfrac{B_{i+i}}{E_i}$.
- It follows that: $TWR = \dfrac{B_2}{E_1} \cdot \dfrac{B_3}{E_2} \cdot \dfrac{B_4}{E_3} \cdot \dfrac{B_5}{E_4} \cdot - 1$

Example 5.10

An investment account is worth 200 at the beginning of the year. Six months later, the account is worth 220, and 120 is withdrawn. Six months after that, the account is worth 85. Find the TWR and DWR during this one year period.

Example 5.11

Assume a fund contains 100 at $t = 0$.
At $t = 1/4$, the fund is worth 120, and 110 is withdrawn.
At $t = 1/2$, the fund is worth 5, and 55 is deposited.
At $t = 1$, the balance of the fund is 75.
Calculated the TWR and DWR.

Example 5.12

On January 1, Perry deposits 150 into an investment fund.
On April 1, the balance of the account is X , and W is withdrawn.
On December 31, the balance of the fund is 140.
The DWR over the 1-year period is 15.69%, and the TWR over the same period is 14.87%.
Find X .

5.4 PORTFOLIO AND INVESTMENT YEAR METHODS

Assume that several people form an investment group, with new members joining periodically. The group makes new investments every year. The effective interest rate earned by any one investment may fluctuate from year to year, and may be different from investments purchased during other years. This section discusses two methods of determining how returns should be distributed to the members.

Portfolio Method

This method ignores when members joined the investment group. At the end of any year, the total return earned on all investments is divided among the members proportionally based on the amount they had invested at the beginning of the year.

Investment Year Method (IYM)

In this method, individual returns are based on the year in which the person joined the group. In any given year, individual contributions will earn returns at different rates, depending upon when the contributions were made.

Let's consider a simple example that will hopefully explain the motivation behind these different methods.

Example 5.13 Mike and Mark start an investment group at the beginning of 2010. They each contribute 1000 and they decide to invest the combined amount of 2000 into Fund A. Over the course of 2010, Fund A earns a return of 6%. The total return is 120, which Mike and Mark split evenly. After they each take their returns of 60, the account still contains 2000 invested in Fund A.

At the beginning of 2011, Judy decides to join the investment group and contributes 1000. Mike and Mark also each contribute an additional 1000. The group decides to invest the 3000 of new money in Fund B. This leaves 2000 in Fund A and 3000 in Fund B.

Over the course of 2011, Fund A earns only a 4% return, while Fund B earns 8%. The total return was $0.04(2000) + 0.08(3000) = 320$. The question now is how to split this return between the three individuals.

Method 1. The total amount invested into the fund is 5000. Mike and Mark each contributed 40% of this amount, whereas Judy contributed only 20% of the total. If we use these percentages to allocate the returns, then Mike and Mark will each get 128 and Judy will receive 64. Under this method, all 3 individuals will earn a 6.4% return during 2011.

Method 2. Judy might argue that her money was only invested in Fund B, and thus she should earn the full 8% on her investment, or 80. That would leave 240 to be split evenly between Mike and Mark, who would each earn 120. Notice that 120 is also equal to the a 4% return on the 1000 each of the two has invested in Fund A, plus a 8% return on the 1000 that they have invested in Fund B. Under this method, Mike and Mark would each earn a total return of 6%, whereas Judy earns 8%.

What would be deemed fair in the previous example probably depends on the perspectives of those involved. Notice that had Fund A done better than Fund B during 2011, Judy would have preferred Method 1.

Reading IYM and Portfolio Rates from a Table

Investment year and portfolio rates are generally reported by using a table, such as the one below. The following comments will explain how to use this table.

- The portfolio rates for a given year are reported in the last two columns. For instance, under the portfolio method, all members earned a 6.2% return during 2011.
- The other entries provide the investment year rates, with each row representing one particular investment year. For instance, someone investing in 2010 would earn 6.4% during 2010, 6.0% during 2011, and 5.7% during 2012.
- For simplicity, it is usually the case that investments are folded into a common portfolio rate once they reach a certain age (three years, in the case of this table).
- As an example, during year 2013, contributions made in 2010 would get folded into the portfolio rate and would thus earn 5.6%. It turns out that all older contributions would also earn this rate of 5.6% in 2013. To see that this is true, notice that a contribution made in 2008 would earn 5.6% in 2008, 6.4% in 2009, 6.3% in 2010. It would then get folded into a portfolio rate making 6.2% in 2011, 5.8% in 2012, and 5.6% in 2013.

Calendar Year of Original Investment	**Investment Year Rates (in %)**			**Portfolio Rates (in %)**	**Calendar Year of Portfolio Rate**
y	i_1^y	i_2^y	i_3^y	i^{y+3}	
2006	5.6	5.8	5.2	5.5	2009
2007	6.0	5.4	6.2	5.8	2010
2008	5.6	6.4	6.3	6.2	2011
2009	6.2	6.5	5.6	5.8	2012
2010	6.4	6.0	5.7	5.6	2013
2011	5.8	5.6	5.4	5.0	2014
2012	5.4	5.6	4.4		
2013	5.2	4.6			
2014	4.8				

Example 5.14

Assume that 1000 is invested in 2009. Find the accumulated value at the end of 2014 using each of the following methods:

a) The portfolio method.
b) The investment year method.
c) Assume that the money is withdrawn at the end of each year, and reinvested at the new money rate.

Example 5.15

Quentin invests 100 at the beginning of each of the years 2010, 2011, 2012, 2013, and 2014. Using the investment year method, determine the accumulated value of Quentin's account at the end of 2014.

CHAPTER 6 – Immunization

6.1 DURATION

Assume we have initiated a project that will yield several cash inflows as well as several cash outflows at various times. Suppose we value the project using an effective annual rate of interest i. For the project to be profitable at this rate, the present value of the cash inflows and outflows must be positive. Interest rates can change over time, resulting in changes in the present value of the cash flows. If interest rates change enough, it is possible that a previously profitable venture will no longer be profitable. In the next few sections, we will study how changing interest rates affect present values.

Price Sensitivity

Assume a bond is priced using an effective yield of i. Now imagine that the next day, the yield rate has changed to $i+\Delta i$, thus changing the value of the bond. The percentage change in the price of the bond resulting from this change in the rate is called the **price sensitivity** of the bond. The price sensitivity of the bond depends strongly on the term of the bond, as we will see in the next example.

Example 6.1

Complete the following problems.

a) Find the prices of 10 and 20 year zero-coupon 1000 par bonds. Assume $i=10\%$.
b) Find the percentage change in the prices of these bonds if the rate changes to $i=9.8\%$.

In the previous example, the price sensitivity of the 20 year bond is roughly twice that of the 10 year bond. That is no coincidence. For zero-coupon bonds, the price sensitivity for a given change in i is approximately proportional to the time until maturity of the bond. The situation for a series of multiple cash flows is a bit more complicated to explain, and requires the introduction of the concept of "duration".

Macaulay Duration

The **Macaulay duration** (or simply **duration**) of a series of cash flows is the time-weighted average of the present values of all of the cash flows. Formulas for the Macaulay duration are given below.

- The Macaulay duration of a general series of cash flows is given by $MacD=\dfrac{\sum\left(t\cdot v^t\cdot CF_t\right)}{\sum\left(v^t\cdot CF_t\right)}=\dfrac{\sum\left(t\cdot v^t\cdot CF_t\right)}{P}$.
- The Macaulay duration of a single cash flow occurring at $t=n$ is $MacD=n$.

Example 6.2

Find the Macaulay duration of a 3-year 100-par bond paying annual coupons of 10% and yielding 8%.

Macaulay Duration and Price Sensitivity

It can be shown that the Macaulay duration of a sequence of cash flows is equal to $-P'(\delta)/P(\delta)$, where $P(\delta)$ is the present value (or price) of the sequence as a function of the force of interest δ. Thus, we can think of $MacD$ as being equal to the price sensitivity resulting from an instantaneous change in δ. However, we are more likely to calculate prices using i than δ. This observation leads us to the definition of "modified duration".

Modified Duration

We define the **modified duration** of a series of cash flows, denoted by $ModD$, to be equal to the ratio $-P'(i) / P(i)$. It can be shown that $ModD = \frac{\sum (t \cdot v^{t+1} \cdot CF_t)}{\sum (v^t \cdot CF_t)}$, or equivalently $ModD = v \cdot MacD$.

Summary of Duration Formulas

We summarize the formulas for Macaulay duration and modified duration below.

- $MacD = \frac{\sum (t \cdot v^t \cdot CF_t)}{\sum (v^t \cdot CF_t)} = \frac{\sum (t \cdot v^t \cdot CF_t)}{P} = -\frac{P'(\delta)}{P(\delta)} = (1+i) ModD$
- $ModD = \frac{\sum (t \cdot v^{t+1} \cdot CF_t)}{\sum (v^t \cdot CF_t)} = \frac{\sum (t \cdot v^{t+1} \cdot CF_t)}{P} = -\frac{P'(i)}{P(i)} = v \cdot MacD$

Example 6.3 Find the modified duration of a 20 year bond paying annual coupons of 50 and maturing for 1000. Assume an annual effective yield of 4%.

Approximating Change in Price

Assume the price of a series of cash flows is equal to P when valued using an effective rate of i. We wish to approximate the change in price ΔP resulting from a change of Δi in the rate. We can rewrite the expression $ModD = -P'(i) / P(i)$ as $P'(i) = -P(i) \cdot ModD$. Since $P'(i) \approx \Delta P / \Delta i$, it follows that $\Delta P \approx -ModD \cdot P \cdot \Delta i$.

Example 6.4 Assuming an annual effective interest rate of $i = 8\%$, an asset stream currently has a present value of 2500. The modified duration of the asset stream is 12.6. Approximate the change in the present value of this stream of payments if the interest rate suddenly increases to $i = 8.5\%$.

Duration of a Perpetuity

The duration of a perpetuity can be calculated in much the same way as any other stream of payments. The primary difference is that the sums involved will now be infinite sums. Consider the following example:

Example 6.5 A perpetuity makes payments of 4 at the end of each year. Assuming an annual effective interest rate of *i*, the perpetuity has a duration of 32.25. Find the price of the perpetuity.

Duration of a Portfolio

Let A and B be two series of payments and let C be a third stream that combines the payments delivered by A and B. The Macaulay duration of C is the price-weighted average of the durations of A and B. In other words:

- $MacD_C = \frac{P_A MacD_A + P_B MacD_B}{P_A + P_B}$

6.2 CONVEXITY

In Section 6.1, we defined modified duration of an asset with specified cash flows in terms of the derivative of the price of that asset with respect to the interest rate. We now define **convexity** in a similar manner, instead using the second derivative of the price with respect to the interest rate. Formulas for convexity are given as follows:

- $Conv = \dfrac{P''(i)}{P} = \dfrac{\sum\left[t\cdot(t+1)\cdot v^{t+2}\cdot CF_t\right]}{P}$

In the following two examples, the summation formula for convexity will probably be the most useful.

Example 6.6 An asset will make payments of 500, 200, and 300 at the end of years 3, 5, and 6, respectively. Assuming an effective annual rate of 6%, calculate the convexity of this asset.

Example 6.7 A 4-year bond pays annual coupons of 6% and has an annual effective yield of 8%. Find the modified duration and the convexity of this bond.

In the next example, the derivative definition of convexity will be the easiest to apply.

Example 6.8 A perpetuity makes payments at the end of each year. The first payment is equal to 5, and subsequent payments increase by 5 per year. Find the modified duration and convexity of this perpetuity, assuming an annual effective yield of 4%.

Approximating Change in Price

In Section 6.1, we used the relationship between $P'(i)$ and modified duration to come up with a first-order approximation for ΔP as a function of Δi. We can use Taylor series to develop a second-order approximation by incorporating convexity into our formula. This yields the following approximation:

- $\Delta P \approx P(i)\cdot\left[-(\Delta i)ModD + \dfrac{1}{2}(\Delta i)^2(Conv)\right]$

Example 6.9 Assuming an annual effective interest rate of $i = 4\%$, an asset stream currently has a present value of 5000. The modified duration of the asset stream is 5 and its convexity is 40. Approximate the change in the present value of this stream of payments if the interest rate suddenly decreases to $i = 3.8\%$.

Convexity of a Portfolio

Let A and B be two series of payments and let C be a third stream that combines the payments delivered by A and B. The convexity of C is the price-weighted average of the convexities of A and B. In other words:

- $$Conv_C = \frac{P_A Conv_A + P_B Conv_B}{P_A + P_B}$$

Example 6.10 Portfolio A has a present value of 320, a duration of 8.75, and a convexity of 80. Portfolio B has a present value of 180, a duration of 12.5, and a convexity of 120. The two portfolios are combined into a single portfolio. Find the duration and convexity of the new portfolio.

6.3 IMMUNIZATION

Assume a portfolio contains several cash inflows as well as several cash outflows. The net present value of such a portfolio is obviously affected by the current effective rates. In fact, the NPV might be positive at one interest rate and negative when calculated using a different rate. The effect that changing rates have on NPV poses a risk to investors and financial institutions. Such entities often employ strategies to minimize their exposure to these interest rate risks. In this section, we will introduce three such methods: Redington immunization, full immunization, and exact matching.

Redington Immunization

A sequence of cash flows is said to be in **Redington immunization** if the following three conditions hold:

1. The PV of the assets equals the PV of the liabilities. That is, $P_A(i) = P_L(i)$.
2. The duration of the assets equals the duration of the liabilities. Equivalently, $P_A{}'(i) = P_L{}'(i)$.
3. The convexity of the assets is greater than the convexity of the liabilities. Equivalently, $P_A{}''(i) > P_L{}''(i)$.

Redington immunization protects the investor from small changes in the interest rate.

The first criteria ensures that the current NPV is zero. The second criteria guarantees that the NPV has a critical point at the current value of *i*. The third criteria results in that critical point being a local minimum for the NPV.

If a set of cash flows satisfies the first two criteria of Redington immunization, it is said to be **duration matched**.

Example 6.11 Two sets of liabilities are given below. Each set of liabilities is duration matched using 2-year and 5-year zero coupon bonds. For each set of liabilities, find the par value of the bond that need to be purchased, and then determine if Redington immunization has been achieved. Assume an annual effective yield of 5%.

a) Liability of 500 at time 1 and another liability of 300 at time 6.
b) Liability of 500 at time 3 and another liability of 300 at time 4.

Example 6.12 A company has a liability portfolio with a present value of 600, a duration of 8, and a convexity of 168. The company plans to duration match its liabilities using the following asset portfolios:

- Portfolio A, which has a duration of 10.25 and a convexity of 210
- Portfolio B, which has a duration of 6.5 and a convexity of *K*.

Find the smallest value of *K* that will achieve Redington immunization.

Full Immunization

A financial enterprise is said to be in **full immunization** if the following three conditions hold:

1. The PV of the assets equals the PV of the liabilities. That is, $P_A(i) = P_L(i)$.
2. The duration of the assets equals the duration of the liabilities. Equivalently, $P_A'(i) = P_L'(i)$.
3. There is one cash inflow before and after each cash outflow. That is, there no two consecutive cashflows that are both liabilities.

Full immunization protects the investor from all changes in the interest rate.

Example 6.13 A liability of 1000 to be repaid at time 6 is fully immunized using an 8-year zero coupon bond and an n-year zero coupon bond. The par value of the 8-year bond is 648.96. The current annual effective interest rate is 4%. Find the par value of the n-year bond.

Example 6.14 BusinessCorp has a liability of 500 due n years from now. They fully immunize the liability by investing in a zero coupon bond that matures for 267 in n – 2 years, as well as a zero coupon bond maturing for 238.2 in $n + t$ years. The current annual effective rate of interest is 6%. Find t.

Exact Matching (Dedication)

Another immunization strategy is to match every liability with an asset to be delivered at the same time and in the same amount as the liability so that there is a net cash flow of 0 at all times. This strategy is called **exact matching** or **dedication**.

Example 6.15 A company has liabilities of 3500 at the end of year 1, 5000 at the end of year 2, and 6500 at the end of year 3. The company exactly matches the liabilities by investing in the following bonds:

i) A one-year zero coupon bond with a yield of 2.5%.
ii) A two-year zero coupon bond with a yield of 3%.
iii) A three-year bond paying annual coupons of 5% and priced to yield 4%.

Find the total amount invested in the three bonds.

Example 6.16 Skyler has liabilities of 2000 due at the end of each of the next three years. She uses dedication to match the liabilities by investing in the following bonds.

i) A one-year bond paying 4% annual coupons.
ii) A two-year bond paying 5% annual coupons.
iii) A three-year bond paying annual coupons at a rate of r.

At an annual effective yield of 6%, the price of the one-year bond was 1716.98. Find r.

CHAPTER 7 – Introduction to Financial Derivatives

7.1 BID AND ASK PRICES

A **market maker** is an entity that facilitates the buying and selling of stocks, or other types of assets. A market maker is generally not interested in making a profit by speculating on the future prices of the assets that they deal will, but instead make money by charging small commissions to any individual using their services to conduct a trade, whether that individual is buying an asset or selling an asset. Some market makers will charge a fixed, or flat, commission for every trade, regardless of the size of the trade, whereas other market makers will set their commissions as a percentage of the value of the overall trade.

When an individual is interested in buying or selling an asset, they submit their request to a market maker along with a price at which they are willing to buy or sell the asset. The market maker than matches compatible buyers and sellers. When looking at the price of an asset on such a service, two prices will be listed: The bid price, and the ask price. These prices are explained below.

- The **bid price** of an asset is the highest price being offered by individuals interested in buying the asset. This is the price at which you could immediately sell the asset.
- The **ask price** of an asset is the lowest price being offered by individuals interested in selling the asset. This is the price at which you could immediately buy the asset.

Note that the listed ask price is always higher than the listed bid price. The difference between the two prices is called the **bid-ask spread.**

Example 7.1

A market maker has 3 offers from individuals wishing to sell a particular asset and 4 offers from individuals wishing to buy the asset. The prices associated with the offers are as follows:

- Sell offers: 30.95, 30.64, 31.24 Buy Offers: 29.97, 30.24, 30.13, 29.83

Determine the bid price, the ask price, and the bid-ask spread.

Example 7.2

Two stocks have bid and ask prices as follows:

- Stock A: Bid price of 60.75, Ask price of 61.50
- Stock B: Bid price of 82.70, Ask price of 83.10

An investor sells two shares of Stock A and buys one share of Stock B. The investor was charged a commission of 0.4% by the market maker. Calculate the investor's net costs.

Example 7.3

On January 1, the bid price of Stock XYZ is 214 and the ask prices is 214.30.
On August 1, the bid price of Stock XYZ is 192.60 and the ask price is 193.
An investor purchased 30 shares of Stock XYZ on January 1, and sold all 30 shares of August 1. The broker charged a flat commission of 15 for each trade. Calculate the investor's net losses, ignoring interest.

7.2 SHORT SALES

A **financial derivative** is an asset whose value is in some way linked to the value of some underlying asset. Derivatives can be bought and sold without either party needing to actually own the underlying asset. There are many different types of derivatives available, some of which increase in value if the underlying asset increases, and some that decrease in value if the underlying asset increases. This leads to the following two definitions.

- An investor is said to hold a **long position** in an asset if they either own the asset outright, or if they own a derivative whose value is positively related to that of the underlying asset. Individuals in a long position hope for the value of the asset to increase.
- An investor is said to hold a **short position** in an asset if they own a derivative whose value is negatively related to that of the underlying asset. Individuals in a short position hope for the value of the asset to decrease.

Short Sales

One of the most common types of derivatives is a short sale. As the name implies, a short sale represents a short position with respect to the underlying asset, which is typically a stock. One would enter a short sale on a stock if they expect the value of the stock to decline. The basic mechanics of a short sale are explained below.

1. The short seller borrows one share of the underlying stock. The lender of the stock is typically a broker. The short seller is required to return the share at some specified time *T*. This is called closing the short.
2. The short seller immediately sells the stock for the current stock price, S_0 .
3. At time *T*, the short seller is required to repurchase the stock at its current price S_T, and then return it to the original lender, thus closing the short.
4. If the stock price has decreased in the interim then $S_T < S_0$ and the short seller will receive a net profit of $S_0 - S_T$, ignoring interest. If the short-seller receives interest on the proceeds of the short sale at an effective annual rate of i, then their profit will be $S_0(1+i)^T - S_T$

Example 7.4

Josh enters a short sale on a stock whose current value is 120. He invests the proceeds from the sale into an account earning an annual effective rate of 3%. He closes the short 4 months later.

a) Determine Josh's net profit if the price of the stock is 105 when the short is closed.
b) Determine Josh's net profit if the price of the stock is 145 when the short is closed.

Risks of Short Selling

When you buy a stock, the most you can lose is the price you paid for the stock. However, there is theoretically no limit to how high the price of a stock could rise during the course of a short sale, and thus there is no limit to the potential losses incurred by a short seller.

Margin

Since there is no limit to the potential price of a stock when it come time to close a short sale, there is a substantial risk to the lender of the stock that the short seller might not be able to afford to repurchase the stock in order to close the short. To offset this risk, the lender often holds the proceeds from the sale of the stock until close, and might also ask for the seller to deposit some amount of collateral in case the price of the stock increases. This collateral is often called **margin** or a **haircut**. The seller generally earns interest on their posted margin, although short sellers do not generally earn interest on the proceeds of the short sale itself.

Example 7.5

Anthony sells a stock short for 8000. The proceeds of the sale are retained by the lender until close (and do not accrue interest). Anthony is required to post margin equal to 75% of the value of the sale. The lender pays interest at an annual effective rate of 4% on the margin account. Anthony closes the short 9 months later. At that time, the price of the stock is 7200. Determine Anthony's annual effective yield on the short sale.

Dividends

Assume that the underlying stock pays a dividend during the course of a short sale. At this point, neither the short seller or the lender of the stock actually owns the stock, and thus neither of the two will receive the dividend. In this scenario, the short seller is generally required to pay the dividend amount to the lender of the stock. Since this transaction typically occurs when the short is closed, it might be necessary for the dividend to be paid back with interest.

Example 7.6

Consider the scenario in Example 7.5. Assume now that the stock pays a dividend of 250 six months after the short sale was initiated. Anthony is required to pay the dividend back at close, along with interest on the dividend, calculated at 4% annual effective. Determine Anthony's annual effective yield.

Example 7.7

Fenton sells a stock short for 1125. The proceeds of the short sale are held by the lender until close, and do not collect interest. Fenton is required to post margin of 80% and is paid interest at an annual effective rate of 5% on his margin account. The short is to be closed exactly one year later. The day before the short is closed, the stock pays a dividend of 20. Fenton's annual effective return on the short sale is -12% . Determine the price of the stock at close.

7.3 FORWARD CONTRACTS

A **forward contract** is an agreement between two individuals in which one party agrees to buy an asset from the other party for a predetermined price on a predetermined date. The price of the asset is decided upon when the contract is entered into, but is not paid until the transaction actually takes place. Some terminology relating to forward contracts is provided below.

- The **expiration date** is the date on which the actual sale will take place.
- The **forward price** is the amount that will be paid for the asset on the expiration date.
- The party obligated to purchase the asset benefits if the value increases, and is thus in a long position with respect to the underlying asset. As such, we say that the buyer has entered into a **long forward**.
- The party obligated to sell the asset benefits if the value decreases, and is thus in a short position with respect to the underlying asset. As such, we say that the seller has entered into a **short forward**.
- A **spot price** is the price of the asset on any specific date (most importantly at expiration).
- The **payoff** to either party involved in a forward contract is the value of the contract to that party on the expiration date. If the forward price is F and the spot price at expiration is S_T, then the payoffs are:
  - **Long Forward Payoff:** $S_T - F$
  - **Short Forward Payoff:** $F - S_T$

Example 7.8 Jack enters into a long forward contract on a stock. If the price of the stock is S at expiration, then Jack's payoff would be 30. If the price of the stock at expiration is $1.2S$, then Jack's payoff would be 46. Determine the forward price.

Cash Settlement

It is often the case that the parties involved in a forward contract will opt for a cash settlement at expiration rather than actually transferring the asset. For instance, if the spot price of the asset at expiration is 120 and the forward price is 100, then the short party might simply pay the long party 20. This is generally done to cut down on transaction fees associated with transferring the asset. When using a cash settlement, the seller is not actually required to own the asset, and might be entering the contract purely for speculative purposes. If the long party actually wants to buy the asset and is not simply engaging in speculation, then they can combine the payoff with the forward price (that never changed hands) and simply buy the asset on the market.

Uses of Forward Contracts

As indicated above, forward contracts are often used for speculative purposes. If you expect the value of an asset to either increase or decrease, then you could enter into a long or short forward as appropriate. Forward contracts are also used as a tool to hedge against potential price increases. If, for instance, a manufacturer knows that they will need to purchase a certain amount of a resource one year from now and suspects that prices for the resource will increase over the next year, then they might use a forward contract to lock in a certain price.

Pricing Forwards on Non-Dividend-Paying Stocks

Two parties entering into forward contract are free to set any forward price that they wish. However, forward prices on stocks are readily bought and sold on the open market, and as a result their prices are set by the market. In fact, if the current risk-free annual effective rate of interest is i and the current value of the stock is S_0, then the price of a forward contract on the stock expiring in T years is equal to $F_{0,T} = S_0(1+i)^T$. When dealing with forward contracts, it is common to see the risk-free rate represented as a continuously compounded rate, r. In this case, the forward price is given by $F_{0,T} = S_0 e^{rT}$.

To understand why forward prices on stocks much be set as stated, consider the following example.

Example 7.9

The current price of a non-dividend stock is 100. The annual effective risk free rate is 6%.

a) Find the correct price of a one-year forward on the stock.

b) Assume that Nikki has an opportunity to enter into a one-year short forward on the stock with a forward price of 107. On the day she enters the forward, she borrows 100 which she uses to buy one share of the stock. She repays the 100 (with interest) on the expiration date for the forward. The price of the stock at expiration is S. Determine Nikki's net cash flows at time 0 and at time 1.

c) Assume that Melvin owns one share of the stock, and finds an opportunity to enter into a one-year long forward on the stock with a forward price of 105. On the day he enters the forward, Melvin sells his share of the stock and invests the proceeds at the current risk-free rate. He withdraws the money at the time of expiration for the forward. The price of the stock at expiration is S. Determine Melvin's net cash flows at time 0 and at time 1.

Arbitrage

The previous example describes two arbitrage scenarios. An **arbitrage** is a set of transactions that allows a party to generate a net positive cash flow at some point in time without any net negative cash flows, and without exposing themselves to any risk. A party who engages in arbitrage is called an **arbitrageur**. Arbitrageurs tend to force prices of forwards on the market to their correct values by capitalizing on any mispriced forwards.

Pricing Forwards on Dividend-Paying Stocks

When pricing a forward on a stock that pays dividends, one must subtract the future value of any dividends paid by the stock from the forward price of the stock. We consider examples involving stocks with discrete dividends, as well as stocks paying continuous dividends. Assume the current price of a stock is S_0 and the continuously compounded risk-free rate of interest is r. Consider a forward on the stock expiring at time *T*.

- If the stock pays discrete dividends, then $F_{0,T} = S_0 e^{rT} - \text{AV}(\text{Divs})$.
- If the stock pays continuous dividends at a rate of δ, then $F_{0,T} = S_0 e^{(r-\delta)T}$.

Example 7.10 The current price of a stock is 120. The stock is expected to pay a dividend of 8 six months from now, and another dividend of 8 one year from now. The continuously compounded risk-free rate of interest is 4%. Find the forward price of a forward contract expiring one year from now, immediately after the second dividend is paid.

Example 7.11 The current price of a stock is 160. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate of interest is 5%. Find the forward price of a 9-month forward contract on the stock.

Example 7.12 The current price of a stock is 230. The stock is expected to pay a dividend of 7 in N months. Assuming a continuously compounded risk-free rate of 4%, the 15 month forward price of the stock is 233.24. Find N.

7.4 PREPAID FORWARD CONTRACTS

A prepaid forward contract is similar to a standard forward contract, except that the buyer pays the seller of the asset when the contract is entered into, as opposed to when the contract is fulfilled. As a result, the prepaid forward price of an asset, denoted by $F_{0,T}^{P}$, is equal to the present value of the forward price of the asset, $F_{0,T}$.

This allows us to consider four possible methods of buying a stock, based on when the payment is received and when the stock is delivered. The four methods are: buying the stock outright, borrowing to pay for the stock, using a prepaid forward, and using a forward contract. The details of these methods are given in the table below.

Method of Buying Stock	Time of Payment	Time Stock is Received	Notation for Pmt	Price (No Div.)	Price (Discrete Div.)	Price (Cont. Div.)
Outright purchase	0	0	--	S_0	S_0	S_0
Borrow to pay for stock	T	0	--	$S_0 e^{rT}$	$S_0 e^{rT}$	$S_0 e^{rT}$
Prepaid Forward contract	0	T	$F_{0,T}^{P}$	S_0	$S_0 - \text{PV}(\text{Divs})$	$S_0 e^{-\delta T}$
Forward contract	T	T	$F_{0,T}$	$S_0 e^{rT}$	$S_0 e^{rT} - \text{AV}(\text{Divs})$	$S_0 e^{(r-\delta)T}$

Example 7.13

The continuously compounded risk-free rate of interest is 6%. Stocks X, Y, and Z all currently sell for 50. Find the price of one-year forward contracts and one-year prepaid forward contracts for each of these three stocks if:

- Stock X pays no dividends.
- Stock Y is scheduled to pay a dividend of 1 in 6 months, and a dividend of 1 in one year (one day prior to expiration of the forward contract).
- Stock Z pays dividends continuously at a rate of 2%.

Example 7.14

A stock has a current price of S and is expected to pay a dividend of 3.5 per share in 6 months. The one-year forward price for the stock is equal to S plus 4.78. The one-year prepaid forward price is 8.15 less than the one-year forward price. Determine the prepaid forward price of the stock.

Forward contracts themselves have inherit value, and can be bought and sold prior to their expiration date. The following example illustrates this concept.

Example 7.15

Assume that the continuously compounded risk-free rate is 4%. On January 1, Stock XYZ has a value of 100. On that day, Gene enters into a two-year long forward contract on XYZ. Six months later, the price of the stock is 92 and Gene enters into an 18 month long forward contract on XYZ. After another six months, the price of the stock is 112. At this point, Gene decides to liquidate his positions in XYZ. Determine the value of each of Gene's forward contracts on this day.

7.5 FUTURES CONTRACTS

A **futures contract** is a very standardized, highly regulated variation to the standard forward contract. Futures and forward are priced in similar ways, but have a few differences. The most important difference between the two instruments is the process of marking-to-market that is undergone by a future.

Marking -to-Market

It is possible for two forward contracts with the same underlying asset and with the same expiration date to have two different market values if they were initiated at different times. This phenomenon was illustrated in Example 7.15. Futures contracts are standardized so that any two futures contracts with the same underlying asset and with the same expiration date will always have the same market value. This is accomplished through a process called **marking-to-market**. Both parties involved in a futures contract must post collateral into an account called a **margin account**. These margin accounts earn interest and are periodically adjusted (usually daily) to reflect changes in the value of the underlying asset. This process guarantees that the market value of the future on any given day is exactly the same as a new future created on that day with the same expiration date.

Additional Characteristics of Futures

Several characteristics common to futures are described below.

- **Notional Value.** A futures contract for a specific asset will generally consist of a fixed number of units of the asset. This quantity is called the notional value of the index.
- **Initial Margin.** To minimize risk of default, both parties involved in a future make a margin deposit. The initial amount of margin required is generally a percentage of the total value of the contract.
- **Maintenance Margin.** Parties are often required to keep a minimum balance in their margin accounts. This minimum is typically a percentage of the initial margin, and is called the maintenance margin.
- **Margin Call.** If a margin account falls below the maintenance margin, the owner is required to make an additional deposit to make up the shortfall. This request for additional margin is called a margin call.

Example 7.16

Stock index XYZ is currently valued at 160. Lex enters into 10 long futures contracts on the index. The notional value for each contract is 200. The initial margin requirement is 10%. Lex's margin account earns interest at a continuously compounded rate of 4% and his position is marked-to-market weekly. The value of the index is 165 at the end of the first week, 175 at the end of the second week, and 170 at the end of the third week. Find the balance of Lex's margin account at the end of the third week.

Example 7.17

Stock index ABC is currently valued at 300. Zoey enters into 20 long futures contracts on the index. The notional value for each contract is 250. The initial margin requirement is 18%, and there is a maintenance margin of 80%. Zoey's margin account earns interest at a continuously compounded rate of 6% and her position is marked-to-market weekly. The value of the index after the first week is X. Find the largest value of X that would result in Zoey receiving a margin call at the end of the first week

CHAPTER 8 – Introduction to Options

8.1 CALL OPTIONS

A **call option** is a type of derivative contract in which the owner of the option has the right, but not the obligation, to purchase the underlying asset for a preset price from the party who sold the option. The details of a call option are explained below.

- The purchaser or holder of the option has the right to decide whether or not to purchase the option at a predetermined time, called the **expiration date**, for a preset price, called the **strike price**.
- If the holder of the the option does decide to purchase the asset when the option expires, the we say that the option has been **exercised**, or that the holder has exercised his right to purchase the asset.
- The individual who sold the option is called the **writer** of the option. The writer of the option is obligated to sell the asset if the purchaser chooses to exercise.
- When buying an option, the purchaser must pay some amount of money to the writer of the option. That amount of money is called the **option premium**. Writers sell options to collect the premium.
- We denote the premium of a call with a strike price of K and with T years until expiration by $\text{Call}(K, T)$.

American vs. European Options

There are two different styles of options in common usage; European and American options. **Europeans options** are only able to be exercised on the date of expiration for the option. **American options**, on the other hand, are able to be exercised on any date up until the expiration date. We will work almost exclusively with European options in this course.

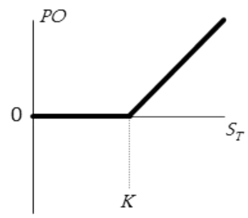
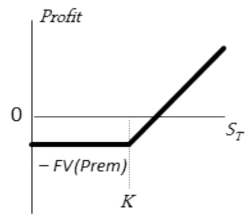
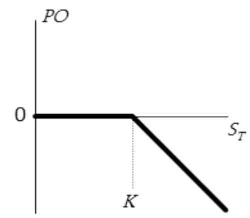
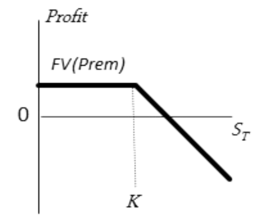
Position with Respect to the Underlying Asset

- **Long (Purchased) Call**. The purchaser of the option hopes that the asset will rise above the strike price so that he or she can purchase the asset at a reduced price. Thus, the purchaser of a call is long with respect to the underlying asset.
- **Short (Written) Call.** The writer of the option hopes that the asset will decrease, reducing the likelihood that the option will be exercised. Thus, the writer of a call is short with respect to the underlying asset.

Call Option Payoff and Profit

The **payoff** of an option for a certain party is the net gain or loss for that party. The **profit** at expiration for an option is the payoff up or down by the future value of the premium, depending on whether the party in question paid or received the premium. In the formulas for payoff and profit provided below, S_T denotes the spot price of the asset at expiration, while K denotes the strike price of the option.

- **Long Call Option:** $PO = \max[0\ ,\ S_T - K]$, $\text{Profit} = \max[0\ ,\ S_T - K] - FV(\text{Prem})$
- **Short Call Option:** $PO = -\max[0\ ,\ S_T - K]$, $\text{Profit} = FV(\text{Prem}) - \max[0\ ,\ S_T - K]$

Long (Purchased) Call		Short (Written) Call	
Payoff	Profit	Payoff	Profit
			

Example 8.1

The current price of one share of GlobalCorp stock is currently 115. Frank purchases a 9-month call on the stock with a strike price of 120. The premium for the call was 11.25. The currently continuously compounded risk-free rate is 4%. Find Frank's payoff and profit at expiration if:

a) The price of the stock at expiration is 150.
b) The price of the stock at expiration is 130.
c) The price of the stock at expiration is 110.

Example 8.2

Crystal purchases two one-year European calls on an asset. One of the calls has a strike price of 80 and a premium of 5.74, and the other call has a strike price of 90 and a premium of 3.29. Assuming a continuously compounded risk-free rate of 3%, Crystal's profit at expiration is equal to 6.695. Find the price of the asset at expiration.

Example 8.3

Doug writes a one-year European call option with a strike price of K and a premium of 15.80. The annual effective risk-free rate of interest is 5%. Doug breaks even on the investment if the spot price at expiration is 178.59. Find K.

Example 8.4

The current price of a stock is $62. Jason makes the following transactions:

- Purchase one 55-strike European call option with a premium of $13.41.
- Write two 60-strike European call options with a premium of $10.46.
- Purchase three 65-strike European call options with a premium of $8.03.
- Write three 70-strike European call options with a premium of $6.06.
- Purchase one 75-strike European call option with a premium of $4.52.

All options above have the same underlying stock and have 1 year until expiration.
The continuously compounded risk-free interest rate is 7%.

Calculate the maximum profit that Jason can obtain from this strategy.

Example 8.5

The spot price of a certain stock is currently \$80.
Grant purchases a one-year 85-strike European call on the stock for a premium of \$8.83.
Heidi writes a one-year 105-strike European call on the same stock for a premium of \$3.37.
The risk-free interest rate is 4%, compounded continuously.
At a spot price of S at expiration, Grant's profit is equal to Heidi's profit. Find S.

Example 8.6

The spot price of a certain stock is currently \$95.
Lori purchases a one-year 100-strike European call on the stock for a premium of \$11.69.
Chad purchases a one-year 120-strike European call on the same stock for a premium of \$5.39.
The risk-free interest rate is 6%, compounded continuously.
At a spot price of S at expiration, Lori's profit is equal to Chads's profit. Find S.

8.2 PUT OPTIONS

A **put option** is similar to a call, except that a put grants the owner of the option the right to sell the option for the strike price at expiration. The terminology relating to put options is directly analogous to that of call options.

- The purchaser or holder of the option has the right to decide whether or not to sell the option at a predetermined time, called the **expiration date**, for a preset price, called the **strike price**.
- If the holder of the the option does decide to sell the asset when the option expires, the we say that the option has been **exercised**, or that the holder has exercised his right to sell the asset.
- The individual who sold the option is called the **writer** of the option. The writer of the option is obligated to buy the asset if the purchaser chooses to exercise.
- When buying an option, the purchaser must pay some amount of money to the writer of the option. That amount of money is called the **option premium**. Writers sell options to collect the premium.
- We denote the premium of a put with a strike price of K and with T years until expiration by $\text{Put}(K, T)$.

Position with Respect to the Underlying Asset

- **Long (Purchased) Put**. The purchaser of a put option benefits from option if the price of the asset drops below the strike price. Thus, the purchaser of a put is short with respect to the underlying asset.
- **Short (Written) Put.** The writer of a put option hopes that the asset will increase, reducing the likelihood that the option will be exercised. Thus, the writer of a put is long with respect to the underlying asset.

Put Option Payoff and Profit

The **payoff** of an option for a certain party is the net gain or loss for that party. The **profit** at expiration for an option is the payoff up or down by the future value of the premium, depending on whether the party in question paid or received the premium. In the formulas for payoff and profit provided below, S_T denotes the spot price of the asset at expiration, while K denotes the strike price of the option.

- **Long Put Option:** $PO = \max[0\ ,\ K - S_T]$, $\text{Profit} = \max[0\ ,\ K - S_T] - FV(\text{Prem})$
- **Short Put Option:** $PO = -\max[0\ ,\ K - S_T]$, $\text{Profit} = FV(\text{Prem}) - \max[0\ ,\ K - S_T]$

<table>
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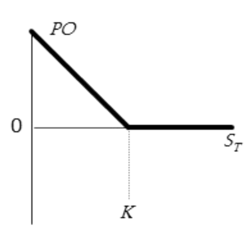
<td>
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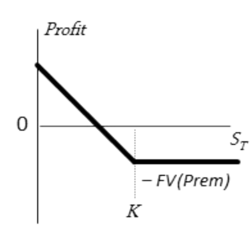
<td>
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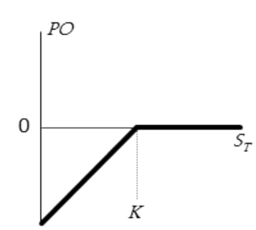
<td>
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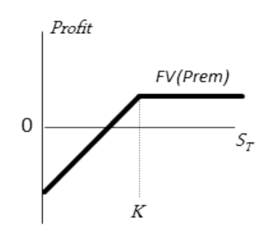
<td>
</td>
</tr>
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Example 8.7

Lillian buys a one-year, 120-strike European put with a premium of \$10.86. The risk free rate of interest is 9.5% effective per annum. At a spot rate of S at expiration, Lillian's profit is 0. Determine S.

Example 8.8

Jonah buys a 6-month 110-strike European put with a premium of \$7.66. He also writes a 6-month 120-strike European put with a premium of \$13.20 on the same underlying asset. The risk-free rate of interest is 6% effective per annum. The spot price at expiration is \$112. Marge's total profit on the two options is X. Find X.

Example 8.9

The spot price of a certain stock is currently \$60.
Tim writes a one-year 50-strike European put on the stock for a premium of \$1.94.
Lars purchases a one-year 70-strike European put on the same stock for a premium of \$11.06.
The risk-free interest rate is 5.5%, compounded continuously.
At a spot price of S at expiration, Tim's profit is equal to Lars's profit.
Find S.

Example 8.10

The spot price of a certain stock is currently \$95.
Anna writes a one-year 90-strike European put on the stock for a premium of \$6.89.
Joan writes a one-year 105-strike European call on the same stock for a premium of \$9.15.
The risk-free interest rate is 4.5%, compounded continuously.
At a spot price of S at expiration, Anna's profit is equal to Joan's profit.
Find S.

8.3 PUT-CALL PARITY

The process of calculating the correct premium for a put or call option is complicated, and is a task that we are not yet ready to fully undertake. However, by comparing the present values of cash flows generated by certain types of derivatives, one can determine a relationship between European put and European call premiums. This relationship is called put-call parity. Three version of this relationship are stated below.

- **General Put-Call Parity:** $\text{Call}(K,T) - \text{Put}(K,T) = PV(F_{0,T}) - PV(K)$
  $\text{Call}(K,T) - \text{Put}(K,T) = F^P_{0,T} - PV(K)$
- **Put-Call Parity for Non-Dividend Stock:** $\text{Call}(K,T) - \text{Put}(K,T) = S_0 - PV(K)$

Note that the put-call parity relationship only holds for European options.

Derivation of Put-Call Parity

Consider two derivative portfolios:

- Portfolio A contains one long call and one short put on an asset. Both options have a strike price of K, and both expire at time T. Convince yourself that regardless of the spot price of the asset at expiration, the owner of this portfolio will end up purchasing the asset for K at time T.
- Portfolio B consists only of a prepaid long forward contract expiring at time T. The underlying asset for the forward is the same as for the options in Portfolio A.

Both portfolios will result in the owner of the portfolio receiving the asset at time T. In order to prevent arbitrage, the costs associated with the two portfolios should have the same present value.

- At time 0, the owner of Portfolio A will pay $Call(K,T)$ and will receive $Put(K,T)$. The owner of Portfolio A will also pay K for the asset at time T. Thus, the present value of the total cost to the owner of Portfolio A is $Call(K,T) - Put(K,T) + PV(K)$.
- The only cash flow for the owner of Portfolio B is a payment of $F^P_{0,T}$ at time 0.
- Setting the present values equal gives us $Call(K,T) - Put(K,T) + PV(K) = F^P_{0,T}$, which can be rewritten as $\text{Call}(K,T) - \text{Put}(K,T) = F^P_{0,T} - PV(K)$.

Example 8.11

The current spot price for a non-dividend-paying stock is \$50. The premium for a 12-month European put with an exercise price of \$55 on that stock is \$6.50. The effective annual interest rate is 8%. Find the price of a 12-month European call option with a strike price of \$55 on the same stock.

Example 8.12

The current forward price for a one-year forward on a certain stock is \$287.92. The premium for a one-year 275-strike European call on the stock is \$38.60, and the premium for a one-year 275-strike European put on the stock is \$26.26. Determine the risk-free annual effective rate of interest.

Example 8.13 The forward price for delivery of one share of XYZ stock in one year is 137.35. The stock does not pay dividends. The continuously compounded risk-free rate of interest is 5.5%. A K-strike one-year European call option on one share of XYZ stock costs 24.17. A K-strike one-year European put option on one share of XYZ stock costs 7.75. Find K.

Hedging Strategies

Occasionally an investor with either a long or short position with respect to a certain asset might wish to enter into an option contract with the opposite position in the underlying asset as a means of providing insurance for the investment. Such a strategy is referred to as **hedging**. Four hedging strategies are described in the table below.

Strategy	Construction	Purpose	Equivalent Strategy (In Terms of Profit)
Protective Put	Long Asset + Long Put	The put places a floor on the amount that the owner is able to sell the asset for.	Long Call
Covered Call	Long Asset + Short Call	The call puts a cap on how much the owner can sell the asset for, but generates a premium that can offset potential losses.	Short Put
Covered Put	Short Asset + Short Put	The put places a floor on the what the short-seller will pay to close, but also generates a premium to offset losses if the price increases.	Short Call
Protective Call	Short Asset + Long Call	The call places a cap on the amount that the short seller is required to pay at close.	Long Put

Example 8.14 Carmen buys a share of stock for $50 and buys a 3-month 50-strike European put at the same time. The premium for a 3-month 50-strike European call is $3.29. The risk-free interest rate is 5% per annum compounded quarterly. Carmen has a profit of 0 at expiration. Find the spot price of the stock at expiration.

Example 8.15 Omar buys a stock for $70 and writes a 70-strike one-year European call on the same stock. The premium for a 70-strike one-year put is $6.34. The risk-free annual effective rate of interest is 5.8%. Find Omar's profit if the spot price at expiration is $77.

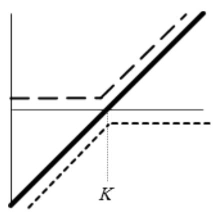
Example 8.16 Geoff sells a stock short for $55 and writes a 3-month European 55-strike put at the same time. The premium for a 3-month 55-strike call is $3.72. The risk-free rate of interest is 6.5% compounded quarterly. Geoff's overall profit is $1. Find the spot price of the asset at expiration.

Example 8.17 Myra sells a stock short for $50 and purchases a one-year European 50-strike call at the same time. The premium for a one-year 50-strike put is $2.62. The risk-free interest rate is 6% effective per annum. The spot price at expiration is $44. Determine Myra's profit.

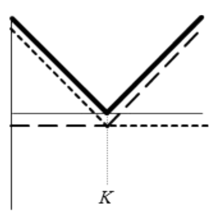
8.4 CONSTRUCTING SPREADS

In this section we will see how to combine puts and calls to construct a variety of financial instruments called spreads. You should be familiar with how to construct these spreads and how to calculate their payoff and profit. You should also understand the strategies for which one might use any particular spread.

Synthetic Forward

	Payoff Graph	Construction	Strategy
Synthetic Forward		• K - Strike Long Call • K - Strike Short Put	• Can be used to create an arbitrage opportunity when there is a mispriced forward contract available.

Straddle

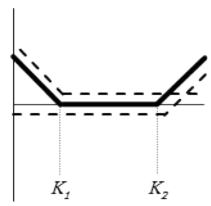
	Payoff Graph	Construction	Strategy
Long Straddle		• K - Strike Long Call • K - Strike Long Put	• This spread is likely to be profitable if the volatility of the underlying asset is high.

A straddle is said to be "at the money" if K is equal to the current price of the stock.

Example 8.18 The price of a certain stock is currently 100. A 100-strike 6-month European call on the stock has a premium of 10.35. A 100-strike 6-month European put on the stock has a premium of 6.50. The nominal risk-free rate of interest is 8%, convertible semiannually. Find the range of prices for which a 6-month at-the-money long straddle is profitable.

Example 8.19 Assuming a spot price at expiration of S, the writer of the straddle in Example 11.9 sees a profit of 8. Find the possible values of S.

Strangle

	Payoff Graph	Construction	Strategy
Long Strangle		• K_1 - Strike Long Put • K_2 - Strike Long Call	• Like a straddle, a strangle is profitable when there are large changes in the value of the asset. • A strangle includes some degree of insurance against stable prices, in that it has a lower potential for loss than a straddle, but also a lower potential for gain. • A strangle has lower financing costs than a straddle.

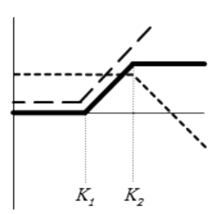
Example 8.20

The following premiums are for one-year European options for an underlying asset with a current spot price of 90.

Strike Price	Call	Put
80	16.99	3.09
90	11.10	6.71
100	6.87	11.99

The continuously compounded risk-free rate of interest is 5%. Determine the range of spot prices for which an at-the-money long straddle has a higher profit than a long 80-100 strangle.

Bull Spread

	Payoff Graph	Construction	Strategy
Bull Spread		• K_1 - Strike Long Call • K_2 - Strike Short Call	• A bull spread is long in the underlying asset. • This spread places a cap on the potential losses, but also places a cap on the maximum potential profit. • An investor might employ this strategy if they expect the asset price to increase only slightly.

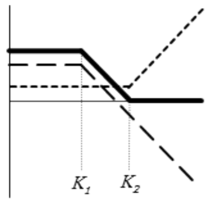
Example 8.21

Assume the conditions described in Example 11.20 still apply. Determine the range of prices at expiration for which an 80-100 bull spread has a positive profit.

Example 8.22

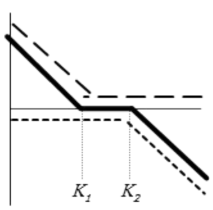
Assume the conditions described in Example 11.20 still apply. Determine the range of prices at expiration for which an 80-100 bull spread has a greater profit than a long 80-100 strangle

Bear Spread

	Payoff Graph	Construction	Strategy
Bear Spread		• K_1 - Strike Short Call • K_2 - Strike Long Call	• A bear spread is short in the underlying asset. • This spread places a cap on the potential losses, but also places a cap on the maximum potential profit. • An investor might employ this strategy if they expect the asset price to decrease only slightly.

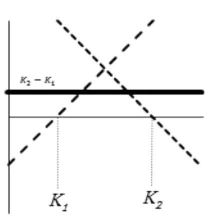
Example 8.23 Assume the conditions described in Example 11.20 still apply. Determine the range of prices at expiration for which an 80-100 bear spread has a greater profit than a written 80-strike call.

Collar

	Payoff Graph	Construction	Strategy
Collar		• K_1 - Strike Long Put • K_2 - Strike Short Call	• A collar is short with respect to the underlying asset. • Can be used to generate a small profit in premiums if the asset price does not change much. • A long or short collar can be combined with an opposing position in the underlying asset to create either a bull or bear spread.

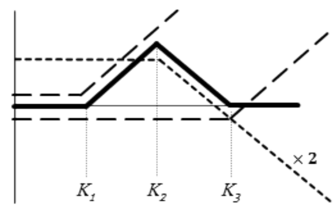
Example 8.24 Assume the conditions described in Example 11.20 still apply. Determine the range of prices at expiration for which an 80-100 written collar is profitable.

Box Spread

	Payoff Graph	Construction	Strategy
Box Spread		• K_1 - Strike Synthetic Fwd • K_2 - Strike Synthetic Fwd	• Generates a guaranteed return at the risk-free rate. • Can be used to take advantage of mispricings in option premiums.

Example 8.25 Assume the conditions described in Example 11.20 still apply. Find the cost of an 80-100 box spread.

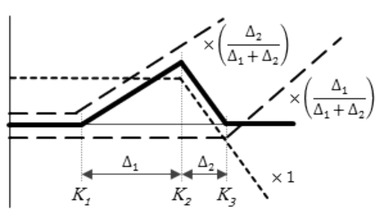
Butterfly Spread

	Payoff Graph	Construction	Strategy
Butterfly Spread		• K_1 - Strike Long Call • K_2 - Strike Short Call (× 2) • K_3 - Strike Long Call or • K_2 - Straddle (Short) • K_1 - K_3 - Strangle (Long)	• Similar to a written straddle, a butterfly spread is profitable when prices are stable. • A butterfly spread caps potential losses, providing insurance against high volatility.

Example 8.26

Assume the conditions described in Example 11.20 still apply. Determine the range of prices for which a written at-the-money straddle generates a higher profit than an 80-90-100 butterfly spread.

Asymmetric Butterfly Spread

	Payoff Graph	Construction	Strategy
Asy. Butterfly		• K_1 - Strike Long Call ($\times \lambda_1$) • K_2 - Strike Short Call (× 1) • K_3 - Strike Long Call ($\times \lambda_2$) $\lambda_1 = \Delta_2/(\Delta_1 + \Delta_2)$ $\lambda_2 = \Delta_1/(\Delta_1 + \Delta_2)$	• The strategies associated with an asymmetric butterfly spread are similar to that of a butterfly spread.

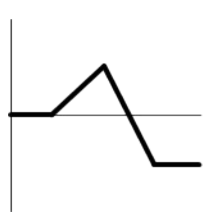
Example 8.27

The table to the right lists premiums for six-month European options for an underlying asset with a current spot price of 55.

An asymmetric butterfly spread is constructed with these options using the smallest whole number of options possible. Determine the range of prices for which this spread is profitable.

Strike Price	Call	Put
50	6.70	1.08
55	3.62	2.94
62	1.21	7.44

Ratio Spread

	Payoff Graph	Construction	Strategy
Ratio Spread		• Any nonstandard mix of calls and puts.	• ?

CHAPTER 9 – Swaps

9.1 COMMODITY SWAPS

A **commodity swap** is a contract that guarantees the delivery of an asset at a set of prescribed times for a certain preset price. It is similar to a forward contract, with the difference being that a forward contract has a single expiration date, whereas a swap is able to cover the several transactions all occurring at different times.

The goal of using a commodity swap to lock in the price of several different transactions could certainly be obtained with several forward contracts, each with a different expiration date and forward price. However, it is often desirable to have the payments involved in the swap contract to all be in the same amount. Such a swap is said to have **level payments**. The present value of the level swap payments must be equal to the present value of the forward prices.

Level Swap Prices

Let $F_{0,k}$ be the current price of a k-year forward on a certain commodity is denoted by . The level swap price L for an n-year swap is found by using the equation: $L\left(v + v^2 + ... + v^n\right) = F_{0,1}v + F_{0,2}v^2 + ... + F_{0,n}v^n$. If interest rates are described using spot rates rather than by an annual effective rate, then the previous equation becomes: $L\left(\frac{1}{1+s_1} + \frac{1}{\left(1+s_2\right)^2} + ... + \frac{1}{\left(1+s_n\right)^n}\right) = \frac{F_{0,1}}{1+s_1} + \frac{F_{0,2}}{\left(1+s_2\right)^2} + ... + \frac{F_{0,n}}{\left(1+s_n\right)^n}$.

Example 9.1

The one, two, and three year forward prices for a certain commodity are currently 60, 64, and 70, respectively. The one, two, and three year sport rates are 5%, 5.4%, and 6%, respectively. Find the level swap price for a 3-year swap that guarantees delivery of one unit of the commodity at the end of each of the next three years.

Implicit Borrowing and Lending

Assume that two parties have entered into a swap contract with level payments. The level payments under the contract will not likely be equal to any of the forward prices for the commodity. As a result it is often assumed that a certain amount of lending and borrowing takes place between the two parties involved in a level swap.

- When the level payment is lower than what would have been expected with a forward, then we assume that the seller is loaning money to the buyer, and that the amount lent is equal to the difference between the forward and swap prices.
- When the level payment is higher than what would have been expected with a forward, then we assume that the buyer is loaning money to the seller, and that the amount lent is equal to the difference between the forward and swap prices.

Since forward prices typically increase as time until expiration increases, the level swap payment will be larger than the forward prices initially, but the swap payments will eventually be smaller than the forward payments. Thus, the buyer is the lending party at the early stages of the swap, but eventually becomes the borrower.

It turns out that the NPV of the amounts borrowed or lent by any party involved in the swap will be zero.

Example 9.2 Determine the implicit amounts borrowed or lent by the buyer in Example 9.1 at the end of each of the first three years. Determine the NPV of the cash flows.

Example 9.3 The one and two-year forward prices of a commodity are 100 and 112, respectively. The two-year level swap price for the commodity is 105.80. The price of a one-year zero-coupon 1000-par bond is currently 943.40. Find the one-year forward rate.

When working with swap, it is generally the case that the swap price will be level throughout the duration of the swap. That does not have to be the case, however. The swap prices could, in theory, be equal to any amounts that the two parties involved agree upon. However, it should always be true that the present value of the swap payments are equal to that of the forward prices.

Example 9.4 The one, two, and three year forward prices for a certain commodity are currently 80, 86, and 92, respectively. The one, two, and three year sport rates are 4.2%, 4.6%, and 5.2%, respectively. A 3-year swap guarantees the delivery of 100 units of the commodity at the end of each of the next three years. The terms of the swap stipulate that the swap price is to increase by 2% each year. Find the total amount paid by the buyer at the end of the first year.

9.2 INTEREST RATE SWAPS

When studying amortization problems in the past, we almost always assumed that the interest rate for the loan was fixed. It is not uncommon, however, for loans to have "floating" rates that change over time. An **interest rate swap** is a contract that replaces a variable or floating rate with a single fixed rate.

Assume that a loan in the amount of 1 is to be repaid at the end of n years. Suppose that interest is to be repaid on the outstanding balance at the end of each year at the current forward rate for that year. Let R be the fixed swap rate for an n-year interest rate swap. Since we are assuming that the amount borrowed is 1, under the swap the loan will be repaid with n annual interest payments of R, followed by a final payment of 1. It follows that 1 is the present value of these payments. That is: $1 = \frac{R}{1+s_1} + \frac{R}{(1+s_2)^2} + \dots + \frac{R}{(1+s_n)^n} + \frac{1}{(1+s_n)^n}$.

Example 9.5 A loan is to be repaid at the end of three year with interest payments due at the end of each year. The size of the interest payments are determined by the forward rates set by LIBOR (London Interbank Offered Rate). The current one, two, and three-year LIBOR spot rates are 5.6%, 5.9%, and 6.4%, respectively. Find the fixed rate for a three-year interest rate swap.

CHAPTER 10 – Binomial Trees

10.1 INTRODUCTION TO BINOMIAL TREES

The payoff for a put or a call is a function of the price of the underlying stock on the expiration date of the option. In order to appropriately price options, we first need construct a probabilistic model for the price of the stock. We can then use this model to determine the probability that a particular option will be exercised, as well as the expected payoff of the option. We can then use this information to price the option.

There are two commonly used stock price models: the binomial tree model, as well as the Black-Scholes framework. In this chapter, we will student the binomial tree model. The basic "one-period binomial tree" model that we start with is a very simplistic model, but we will see later that it serves as the building block for more complicated and more realistic stock models.

One-Period Binomial Trees

A one-step binomial tree model is described as follows.

- Let S be the current price of the stock. This will sometimes be denoted by S_0 .
- The model covers a specific length of time. The period length is denoted by h, and is measured in years.
- We assume that there are only two possible values of the stock at time h. Either the price of the stock will increase to a value of S_u , or it will decrease to a value of S_d .
- The values S_u and S_d are sometimes stated explicitly, but are often provided as multiples of S. If the multipliers u and d are provided, then $S_u = S \cdot u$ and $S_d = S \cdot d$.
- The probability of an up-move is denoted by p . The probability of a down-move is equal to $q = 1 - p$.

Expected Stock Price and Expected Annual Return

Given a one-period binomial tree, we may calculate the following.

- The **expected price** of the stock after 1 period is $E\left[S_h\right] = p \cdot S \cdot u + q \cdot S \cdot d$.
- The **capital gains rate** g is the continuously compounded annual rate of growth that would cause the initial stock price of S to grow to the expected value $E\left[S_h\right]$. In other words, $S\,e^{gh} = E\left[S_h\right]$. It follows that $g = \frac{1}{h} \cdot \ln\left(\frac{E\left[S_h\right]}{S}\right) = \frac{\ln\left(p \cdot u + q \cdot d\right)}{h}$.
- The **continuously compounded expected annual rate of dividend growth** for the stock is denoted by δ . We will consider only non-dividend paying stocks in this section.
- The **continuously compounded expected annual rate of return** α for the stock is equal to the capital gains rate plus the continuously compounded expected rate of dividend grown, δ . That is, $\alpha = g + \delta$.

Notice that the expected rate of return α is the expected return on investment for someone purchasing the stock. They would expect return of g due to price changes, as well as an additional return of δ due to dividend growth. This gives an total return of $\alpha = g + \delta$. For a risk-averse investor to be interested in a particular stock, they would require α to be larger than the risk-free rate r .

Example 10.1 The price of a non-dividend-paying stock currently worth 50 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 8 months. The probability of an up-move is $p = 0.65$.

a) Determine the expected price of the stock after 8 months.

b) Find the continuously compounded expected annual return for the stock over the 8 month period.

Example 10.2 The price of a non-dividend-paying stock currently worth 100 is modeled by a one-period binomial tree with $u = 1.15$ and $d = 0.9$. The period for the tree is 9 months. The continuously compounded expected annual return for the stock is $\alpha = 8\%$. Determine the probability of an up-move.

Expected Payoff and Return for Options

Given a binomial tree model for a stock, we can use the model to determine the expected payoff as well as the expected return for a call or put on that stock expiring at the end of the period. The details are provided below.

- Consider a *K*-strike European call and a *K*-strike European put on a stock, both expiring at the end of one period. We will denote the payoff of the call at the up-node by C_u, and the payoff of the call at the down-node by C_d. Similarly, we will denote the payoff of the put at the up and down nodes by P_u and P_d, respectively. Formulas for these quantities are given by:
  - $C_u = \max\left[0,\, S_u - K\right]$ and $C_d = \max\left[0,\, S_d - K\right]$
  - $P_u = \max\left[0,\, K - S_u\right]$ and $C_d = \max\left[0,\, K - S_d\right]$
- The expected payoffs of the options are given by $E\left[\text{Call } PO\right] = p\,C_u + q\,C_d$ and $E\left[\text{Put } PO\right] = p\,P_u + q\,P_d$.
- The continuously compounded expected annual return for a particular option is denoted by γ. It is defined by $\text{Premium} \cdot e^{\gamma h} = E\left[\text{Option } PO\right]$, or $\gamma = \dfrac{1}{h} \ln\left(\dfrac{E\left[\text{Option } PO\right]}{\text{Premium}}\right)$.

Example 10.3 The price of a non-dividend-paying stock currently worth 50 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 8 months. The probability of an up-move is $p = 0.65$. An 8-month 112-strike call has a premium of 4.42. The continuously compounded risk-free rate is 4%.

a) Determine the expected call payoff and the expected return for the call.

b) Determine the expected put payoff and the expected return for the put.

Example 10.4 The price of a non-dividend-paying stock currently worth 120 is modeled by a one-period binomial tree $u = 1.1$ and $d = 0.9$. The period for the tree is 6 months. A 6-month 125-strike put has a premium of 6.1915 and an expected yield of 8.4911%. Find the expected return for the stock.

10.2 REPLICATING PORTFOLIOS

Given a binomial tree for a stock, it is not difficult to calculate the expected payoff for an option on that stock. The premium for the option should then be the present value of this expected payoff. The issue we face is that the rate that we should use to discount the expected payoff is γ, the expected yield of the option. If we don't already know the premium, we don't have a method of calculating γ. This requires us to develop alternate methods for pricing options. We will learn two methods: replicating portfolios and risk-neutral pricing. We will cover replicating portfolios in this section, and risk-neutral pricing in the next.

Replicating Portfolios

The method of replicating portfolios allows us to price options on a stock modeled by a binomial tree without using any probabilistic concepts. In this method, we will construct a portfolio consisting of some shares of the underlying asset, as well as some amount of borrowing of lending. The portfolio will be built so that the payoffs at time h are exactly the same as the option in consideration at both the upper and lower nodes. Since those are the only two "possible" payoffs for the option in the binomial tree model, we conclude that the price of the option must be the same as the price of our replicating portfolio.

The process of pricing an option using a replicating portfolio is outlined below.

- Assume a stock is modeled using a 1 period binomial tree with S, r, δ, h, u, and d given.
- We construct a portfolio by buying Δ shares of the stock, and investing (lending) B in risk-free bonds.
- The cost of our replicating portfolio is $\Delta S + B$.
- The portfolios value at $t = h$ is $\Delta e^{\delta h} Su + Be^{rh}$ at the up node and $\Delta e^{\delta h} Sd + Be^{rh}$ at the down node.
- Let V_u and V_d be the payoffs for the option being priced at the upper and lower nodes respectively.
- For either a call or a put, we set $\Delta e^{\delta h} Su + Be^{rh} = V_u$ and $\Delta e^{\delta h} Sd + Be^{rh} = V_d$. Solving this system yields $\Delta = \left(\dfrac{V_u - V_d}{Su - Sd}\right) e^{-\delta h}$ and $B = \left(\dfrac{uV_d - dV_u}{u - d}\right) e^{-rh}$.
- The price of the option is then $V = \Delta S + B$.
- Note that for a call, $\Delta \geq 0$ and $B \leq 0$. In contrast, for a put we have $\Delta \leq 0$ and $B \geq 0$.
- Let Δ_C be the number of shares in the replicating portfolio for a K-strike call and let Δ_P be the number of shares in the replicating portfolio for a K-strike put. A consequence of the previous derivations is that $\Delta_C - \Delta_P = e^{-\delta h}$.

We begin by looking at a few examples involving non-dividend-paying stocks.

Example 10.5

The price of a non-dividend-paying stock currently worth 130 is modeled by a one-period binomial tree $u = 1.2$ and $d = 0.8$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4.5%.

a) Calculate the premium of a one-year 128-strike call on the stock.

b) Calculate the premium of a one-year 128-strike put on the stock.

Example 10.6 The price of a non-dividend-paying stock currently worth 120 is modeled by a one-period binomial tree $u = 1.3$ and $d = 0.85$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4%. Find the strike price of a one-year call option whose replicating portfolio contains 0.5926 shares of the stock.

We will now consider an example involving dividend-paying stocks.

Example 10.7 The price of a stock currently worth 100 is modeled by a one-period binomial tree $u = 1.3$ and $d = 0.8$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 5%.

c) Calculate the premium of a one-year 98-strike call on the stock.

d) Calculate the premium of a one-year 98-strike put on the stock.

Example 10.8 The price of a stock is modeled using a one-period binomial tree with a period of six months. The difference between the price of the stock at the upper and lower nodes is 48. The difference between the payoffs of a six-month K-strike call at the upper and lower nodes is 31. The stock pays continuous dividends at a rate of 3%. Find the number of shares in the replicating portfolio for a six-month K-strike put on the stock.

10.3 RISK NEUTRAL PRICING

The method of risk neutral pricing provides an alternative to replicating portfolios for pricing options on stocks whose prices are modeled using binomial trees. The two methods produce the same results, but each have their own advantages. For some applications, the value of delta found using the replicating portfolio method might be of interest in its own rate. An advantage of risk neutral pricing is that it tends to be easier to apply when working with a multi-period binomial tree.

Risk Neutral Pricing Method

When using risk neutral pricing, we assume that we are in a "risk-neutral" world where every investment is expected to grow at the risk-free rate. The details of the method are explained below.

- Assume that the underlying stock is modeled by a one period binomial tree with parameters S, u, d, δ, and h. Also assume that the risk-free rate r is given.
- To use the risk neutral model, we do not need to know the value of p. Recall that if we did know p, then we could be able to calculate the expected price of the stock $E[S_h] = p \cdot S \cdot u + q \cdot S \cdot d$. We could subsequently calculate the capital gains rate g using $S e^{gh} = E[S_h]$, and the expected yield $\alpha = g + \delta$.
- When using risk neutral pricing, we assume that the expected yield is equal to r and then calculate the **risk neutral probability** p^* consistent with this return. This amounts to solving for p^* in the equation $p^* \cdot S \cdot u + (1 - p^*) \cdot S \cdot d = S e^{(r-\delta)h}$.
- The quantity $E^*[S_h] = p^* \cdot S \cdot u + (1 - p^*) \cdot S \cdot d$ is called the **risk neutral expected value of the stock**.
- Solving for p^* gives the formula $p^* = \dfrac{e^{(r-\delta)h} - d}{u - d}$.
- Assume now that we wish to price an option that has values of V_u and V_d at the up and downs. The **risk neutral expected payoff** of the option is given by $E^*[PO] = p^* V_u + (1 - p^*) V_d$.
- The premium is then obtained by discounting the risk neutral expected payoff using the risk-free rate. That is: $\text{Premium} = \left[p^* V_u + (1 - p^*) V_d \right] e^{-rh}$.

Example 10.9

The price of a stock currently worth 100 is modeled by a one-period binomial tree $u = 1.3$ and $d = 0.8$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 5%.

a) Use risk neutral pricing to price a one-year 98-strike call on the stock.

b) Use risk neutral pricing to price a one-year 98-strike put on the stock.

The problem we just solved is identical to the one presented in Example 10.7. Compare the results to verify that the two methods do in fact generate the same prices.

Example 10.10

The price of a stock currently worth 140 is modeled by a one-period binomial tree $u = 1.25$ and $d = 0.85$. The probability of an up-move is $p = 0.4$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 4.5%. Find the expected yield for a one-year 135-strike European put on the stock.

10.4 MULTI-PERIOD BINOMIAL TREES

As mentioned previously, one-period binomial trees are not a particularly realistic model for stock prices. We can obtain more realistic models by expanding upon the idea and considering multi-period binomial trees. The details of the multi-period binomial tree model are explained below.

- Assume that the length of time covered by the model is T years, and that this interval of time is split into n periods of length h.
- The initial stock price is S. Prices at later nodes are denoted using subscripts indicating the number of up and down moves required to reach that node.
- The probability of an up-move for any given period is p. In the case of an up-move, the price is multiplied by a factor of u. The multiplier for a down-move is d.

Pricing European Options Using Multi-Period Binomial Trees

We will explain how to price a European Option using a two-period binomial tree. The process for using a binomial tree with more than two periods is a natural extension to this method.

1. Assume that we are pricing a $2h$-year K-strike European option.
2. Denote the payoffs of the option at each of the three terminal nodes by V_{uu}, V_{ud}, and V_{dd}.
3. Use the payoffs V_{uu} and V_{ud} to calculate the price of a h-year K-strike option of the same type, sold at time h, assuming that an up-move occurred during the first period. Denote this quantity by V_u.
4. Use the payoffs V_{ud} and V_{dd} to calculate the price of a h-year K-strike option of the same type, sold at time h, assuming that a down-move occurred during the first period. Denote this quantity by V_d.
5. Use the values V_u and V_d to determine the price of the option, V.

An alternate (and equivalent) method would be to calculate the risk-neutral expected payoff at time $2h$ using $E^*[S_{2h}] = (p^*)^2 S_{uu} + 2\,p^* q^* S_{ud} + (q^*)^2 S_{dd}$, and then discount to time 0 using the risk-free rate: $V = E^*[S_{2h}]e^{-2rh}$.

Example 10.11

The price of a stock currently worth 160 is modeled by a two-period binomial tree $u = 1.2$ and $d = 0.8$. Each period is 6 months. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 6%.

a) Calculate the premium for a one-year 185-strike European call on the stock.

b) Calculate the premium for a one-year 185-strike European put on the stock.

Pricing American Options Using Multi-Period Binomial Trees

Recall that American options and European options differ in that European options can only be exercised when the option expires, whereas an American option can be exercised at any point prior to the expiration date for the option. We can use multi-period binomial trees to price American options by making a small adjustment to the process using for European options. We explain the process for a two period binomial tree below.

1. Assume that we are pricing a $2h$-year K-strike American option. We also assume that the option in question can only be exercised at the end of an h-year period.

2. Denote the payoffs of the option at each of the three terminal nodes by V_{uu}, V_{ud}, and V_{dd}.

3. We now calculate V_u. The process is more complicated than with European options.
   a) Use the payoffs V_{uu} and V_{ud} to find the price of a h-year K-strike option of the same type, sold at time h, assuming that an up-move occurred during the first period. Denote this quantity by MV_u.
   b) Find the payoff for the option at the up-node if it were exercised early. Denote this by PO_u.
   c) Let $V_u = \max\left[MV_u, PO_u\right]$.

4. We now calculate V_u.
   a) Use the payoffs V_{ud} and V_{dd} to find the price of a h-year K-strike option of the same type, sold at time h, assuming that a down-move occurred during the first period. Denote this quantity by MV_d.
   b) Find the payoff for the option at the down-node if it were exercised early. Denote this by PO_d.
   c) Let $V_d = \max\left[MV_d, PO_d\right]$.

5. Use the values V_u and V_d to determine the price of the option, V.

Example 10.12

The price of a stock currently worth 160 is modeled by a two-period binomial tree $u = 1.2$ and $d = 0.8$. Each period is 6 months. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 6%.

a) Calculate the premium for a one-year 185-strike American call on the stock.

b) Calculate the premium for a one-year 185-strike American put on the stock.