CHAPTER 1 – Forwards and Options

1.1 FORWARD CONTRACTS

Forward Contracts

A **forward contract** is an agreement between two individuals in which one party agrees to buy an asset from the other party for a predetermined price on a predetermined date. The price of the asset is decided upon when the contract is entered into, but is not paid until the transaction actually takes place. Some terminology relating to forward contracts is provided below.

- The **expiration date** is the date on which the actual sale will take place.
- The **forward price** is the amount that will be paid for the asset on the expiration date.
- The party obligated to purchase the asset benefits if the value increases, and is thus in a long position with respect to the underlying asset. As such, we say that the buyer has entered into a **long forward**.
- The party obligated to sell the asset benefits if the value decreases, and is thus in a short position with respect to the underlying asset. As such, we say that the seller has entered into a **short forward**.
- A **spot price** is the price of the asset on any specific date (most importantly at expiration).
- The **payoff** to either party involved in a forward contract is the value of the contract to that party on the expiration date. If the forward price is $F_{0,T}$ and the spot price at expiration is S_T, then the payoffs are:
  - **Long Forward Payoff:** $S_T - F_{0,T}$
  - **Short Forward Payoff:** $F_{0,T} - S_T$

Prepaid Forward Contracts

A prepaid forward contract is similar to a standard forward contract, except that the buyer pays the seller of the asset when the contract is entered into, as opposed to when the contract is fulfilled. As a result, the prepaid forward price of an asset, denoted by $F_{0,T}^P$, is equal to the present value of the forward price of the asset, $F_{0,T}$.

Forward Prices

The economic "Law of one Price" can be used to find formulas for arbitrage-free forward and prepaid forward prices. The following table summarized these prices under a variety of circumstances.

	$F_{0,T}$	$F_{0,T}^P$
Stock does not pay dividends	$S_0 e^{rT}$	S_0
Stock pays discrete dividends	$S_0 e^{rT} - \text{AV}(\text{Divs})$	$S_0 - \text{PV}(\text{Divs})$
Stock pays dividends continuously at a rate of δ	$S_0 e^{(r-\delta)T}$	$S_0 e^{-\delta T}$

Notice that in every row in the table above, we have that $F_{0,T}^P = PV\left(F_{0,T}\right)$.

Forwards on Currency

Forward contracts are occasionally used to lock in exchange rates of future currency exchanges. The formulas involved with working with currency forwards are no different from other forwards, but do tend to require some symbolic substitutions. The details are explained below.

- Imagine that we intend to use a domestic currency to purchase one unit of a foreign currency at time T.
- For convenience, denote the two currencies by $\$_d$ and $\$_f$.
- Since we will be "buying" the foreign currency with our domestic currency, it is helpful to think of the unit of foreign currency as the underlying asset.
- We will denote the current exchange rate by x_0. This is the current cost of one unit of the foreign currency, expressed in units of the domestic currency. In other words, $\$_f 1 = \$_d x_0$ today.
- Assume that the domestic currency earns interest at a risk-free rate of r_d. Since this is the rate at which our money grows, this will play the role of the normal risk-free rate in our previous formulas.
- Assume that the foreign currency earns interest at a risk-free rate of r_f. This is the rate at which our "asset" grows, and will thus take the place of the dividend rate in the previous formulas.
- To determine the forward or prepaid forward price of a currency exchange, we can use the continuous dividend formulas along with the following substitutions: $S_0 = x_0$, $r = r_d$, and $\delta = r_f$.

Example 1.1

The currency exchange rate between euros and dollars is currently $€1 = \$1.14$. Assume that the current risk-free rate for dollars is 4% and the current risk-free rate for euros is 1%. Determine the forward price for a forward contract that guarantees the delivery of $€1000$ five years from now.

1.2 INTRODUCTION TO OPTIONS

Calls and Puts

An option is a derivative instrument that allows the owner of the option the right to "exercise" the option at a certain point in time in order to receive a payoff that is somehow determined by the price of the underlying asset at that moment. Their are many types of options, but the fundamental options are calls and puts.

- A **call option** is a type of derivative contract in which the owner of the option has the right, but not the obligation, to purchase the underlying asset for a preset price from the party who sold the option.
- A **put option** is a type of derivative contract in which the owner of the option has the right, but not the obligation, to sell the underlying asset for a preset price to the party who sold the option.

Characteristics of an Option

The following list defines several terms relevant to options.

- The purchaser or holder of the option has the right to decide whether or not to purchase the option at a predetermined time, called the **expiration date**, for a preset price, called the **strike price**.
- If the holder of the the option does decide to purchase the asset when the option expires, the we say that the option has been **exercised**, or that the holder has exercised his right to purchase the asset.
- The individual who sold the option is called the **writer** of the option. The writer of the option is obligated to sell the asset if the purchaser chooses to exercise.
- When buying an option, the purchaser must pay some amount of money to the writer of the option. That amount of money is called the **option premium**. Writers sell options to collect the premium.
- We will often denote the premium of a call as either "Call" or "C". Similarly, we will generally denote the premium of a put as either "Put" or "P". When working with an arbitrary option that could be either a call or a put, we will usually denote its premium by "V".

American vs. European Options

There are two different styles of options in common usage; European options and American options. **Europeans options** are only able to be exercised on the date of expiration for the option. **American options**, on the other hand, are able to be exercised on any date up until the expiration date.

Position with Respect to the Underlying Asset

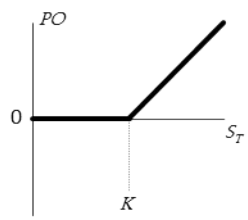
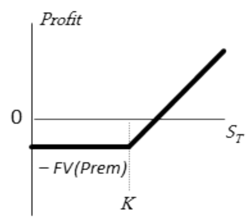
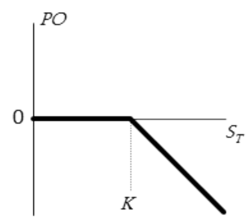
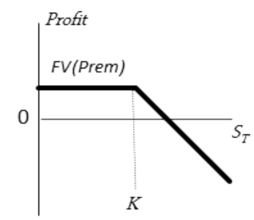
The two parties involved in an option will hold opposite positions with respect to the underlying asset, as explained below.

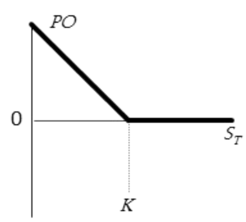
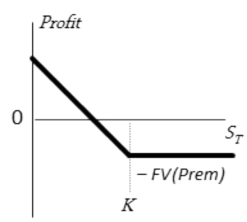
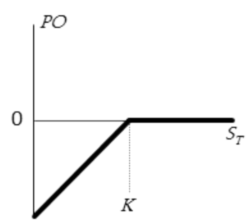
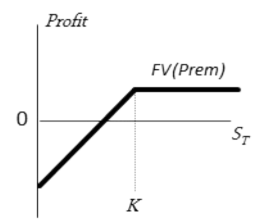
- **Long (Purchased) Call**. The purchaser of a call is long with respect to the underlying asset.
- **Short (Written) Call.** The writer of a call is short with respect to the underlying asset.
- **Long (Purchased) Put**. The purchaser of a put is short with respect to the underlying asset.
- **Short (Written) Put.** The writer of a put is long with respect to the underlying asset.

Option Payoff and Profit

The **payoff** of an option for a certain party is the net gain or loss for that party. The **profit** at expiration for an option is the payoff up or down by the future value of the premium, depending on whether the party in question paid or received the premium. In the formulas for payoff and profit provided below, S_T denotes the spot price of the asset at expiration, while K denotes the strike price of the option.

- **Long Call Option:** $PO = \max\left[0\ ,\ S_T - K\right]$, Profit $= \max\left[0\ ,\ S_T - K\right] - FV\left(\text{Prem}\right)$
- **Short Call Option:** $PO = -\max\left[0\ ,\ S_T - K\right]$, Profit $= FV\left(\text{Prem}\right) - \max\left[0\ ,\ S_T - K\right]$
- **Long Put Option:** $PO = \max\left[0\ ,\ K - S_T\right]$, Profit $= \max\left[0\ ,\ K - S_T\right] - FV\left(\text{Prem}\right)$
- **Short Put Option:** $PO = -\max\left[0\ ,\ K - S_T\right]$, Profit $= FV\left(\text{Prem}\right) - \max\left[0\ ,\ K - S_T\right]$

Long (Purchased) Call		Short (Written) Call	
Payoff	Profit	Payoff	Profit
			

Long (Purchased) Put		Short (Written) Put	
Payoff	Profit	Payoff	Profit
			

Option Pricing

The primary goal of this course is to learn how to price options, or in other words, to calculate their premiums. Since options derive their value from some underlying asset, which is usually a stock, to be able to price an option, we much first develop a probabilistic model for the price of the underlying asset. We will work with two common stock models in this course: The binomial tree model, and the lognormal model. Each model has its advantages under certain circumstances.

As we will see later, the price of an option is usually represented as a function of the following six variables:

- Current stock price, denoted by S_0, or occasionally just S.
- The strike price, K.
- The time until expiration, T.
- The continuously compounded risk-free rate of interest, r.
- The continuously compounded dividend rate for the underlying asset, δ.
- The volatility in the price of the underlying asset, denoted by σ.

1.3 PUT – CALL PARITY

Put-Call Parity Formula

The premiums of a European call and a European with the same underlying stock and the same characteristics satisfy an important relation called put-call parity. Three versions of this relationship are stated below.

- **General Put-Call Parity.** $C(K,T) - P(K,T) = PV(F_{0,T}) - PV(K)$
  $C(K,T) - P(K,T) = F_{0,T}^{P} - PV(K)$
- **Put-Call Parity for Non-Dividend Stock:** $C(K,T) - P(K,T) = S_0 - PV(K)$

Note that the put-call parity relationship only holds for European options.

The derivation for the put-call parity relation can be found in the FM notes. We will reproduce it here, but it would be useful to review. It is important to note that the parity relation must hold regardless of the model used to price the options.

Alternate Notation. he formula for put-call parity is often written as $C(K,\ T) - P(K,\ T) = F^{P}(S) - F^{P}(K)$. Here $F^{P}(S)$ is the prepaid forward price of the stock and $F^{P}(K)$ is the prepaid forward price of a future payment of *K*, or in other words, the price of a zero-coupon bond maturing for *K*.

Example 1.2

Four portfolios are constructed using 2-year European options, all on the same stock.

- Portfolio A consists of a long 70-strike call and a short 75-strike put. Its cost is 13.03.
- Portfolio B consists of a long 70-strike put and a short 75-strike call. Its cost is -12.31.
- Portfolio C consists of a long 85-strike call and a short 90-strike put. Its cost is -1.71.
- Portfolio D consists of a long 85-strike put and a short 90-strike call. Its cost is 0.81.

Determine the continuously compounded risk-free rate of interest.

Example 1.3

The price of a stock is currently 80. The stock pays dividends continuously at a rate of 2%. You observe on the market a one-year 90-strike European call on the stock with a premium of 10.81. You also observe a one-year 90-strike European put on the same stock with a premium of 14.84. The continuously compounded risk-free rate of interest is 6%.

You determine that an opportunity for arbitrage exists. Explain how to construct the arbitrage and give the present value of the potential arbitrage profit per share of the stock.

Example 1.4

An investor is considering several *T*-year options, all on the same stock.
Let $f(K)$ denote the premium of a *K*-strike call minus the premium of a *K*+1-strike call.
Let $g(K)$ denote the premium of a *K*-strike put minus the premium of a *K*+1-strike put.
The investor notes that $f(80) = 0.54$, $g(80) = -0.38$, and $g(100) = -0.63$. Find $f(100)$.

Currency Options

In Section 1.1 we considered forward contracts on currency exchanges. It is also possible to create calls and puts on currency exchanges. The details are similar to forwards: You should think of the currency being purchased as the asset, with a current price of x_o. That currency's risk-free rate plays the role of the dividend rate, while the strike currency's risk-free rate plays the role of the risk-free rate.

When working with currency options, you will often see phrases such as “A dollar-denominated put on yen.” The denomination of an option is the currency that the strike is paid in, and (usually) the currency in which the premium is paid.

Example 1.5

A one-year dollar denominated European call on pounds with a strike price of 1.60 currently sells for \$0.08. A one-year dollar denominated European put on pounds with a strike price of 1.60 currently sells for \$0.02. The continuously compounded risk-free rate for dollars is 2%, while the continuously compounded risk-free rate for pounds is 5%. Determine the current exchange rate in dollars per pound.

Option Duality

Let X and Y each represent assets. They could represent money, stocks, bonds, or any other sort of asset. They could both represent amounts of money, but perhaps in different currencies. Assume that an option allows the holder of the option to give up Asset X in exchange for Asset Y at expiration. This option could be viewed in two different ways:

- It can be seen as a call in which Asset X represents the strike price and Asset Y represents the underlying asset. We often denote calls such as this as $C(A=Y\,,\ K=X)$.
- It can be seen as a put in which Asset X represents the underlying asset and Asset Y represents the strike price. We often denote puts such as this as $P(A=X\,,\ K=Y)$.

We will study options in which the strike asset and underlying asset are both stocks (called exchange options) in some detail in Section 5.6. For now, note that since $C(A=Y\,,\ K=X)$ and $P(A=X\,,\ K=Y)$ are two ways of denoting the same option, it must be the case that $C(A=Y\,,\ K=X) = P(A=X\,,\ K=Y)$. This observation can be useful when working with options on currency exchanges.

Example 1.6

The continuously compounded risk-free rates for dollars and yen are currently 4% and 3%, respectively. A four-year yen-denominated European call on \$1 with a strike price of ¥125 currently has a premium of ¥2.2793. A four-year dollar-denominated European call on ¥1 with a strike price of \$0.008 currently has a premium of \$0.0014115. Determine the current exchange rate in yen per dollar.

Example 1.7

A financial institution knows that it will need to purchase €200,000 two years from now in order to satisfy certain obligations. The company decides to hedge against rising exchange rates and purchases options allowing them to pay no more than \$250,000 for the euros. Find the price of these options, in dollars, given the following information:

- The current exchange rate is \$1.15 per €1.
- The continuously compounded risk-free rate for dollars is 4%.
- The continuously compounded risk-free rate for euros is 3%.
- A two-year European call on \$1 with a strike of €0.80 currently costs €0.08.

1.4 BOUNDS FOR OPTIONS PREMIUMS

We will soon begin to learn methods for pricing options. In this section, however, we will discuss certain bounds that option premiums must adhere to regardless of the pricing model used to generate them.

Let C_{EUR}, P_{EUR}, C_{AM}, and P_{AM} denote the premiums for a European call, a European put, an American call, and an American put on the same underlying asset, which is currently valued at *S*. Assume all four options have a strike price of *K*, and all four expire at time *T*. The following inequalities provide bounds for these premiums.

Bounds for Option Premiums

1. C_{EUR}, P_{EUR}, C_{AM}, and P_{AM} are all greater than or equal to 0.
2. $P_{EUR} \le P_{AM}$ and $C_{EUR} \le C_{AM}$
3. $S e^{-\delta T} - K e^{-rT} \le C_{EUR} \le S e^{-\delta T}$ and $S - K \le C_{AM} \le S$
4. $K e^{-rT} - S e^{-\delta T} \le P_{EUR} \le K e^{-rT}$ and $K - S \le P_{AM} \le K$

Brief explanations for the inequalities above are given below.

1. Is it never harmful to own an option, and so an option can never have a negative premium.
2. American options carry all of the rights of European options, and thus and American option will never be cheaper than a similar European option.
3. The inequality $S e^{-\delta T} - K e^{-rT} \le C_{EUR}$ follows from put-call parity, along with the observation that $P_{EUR} \ge 0$. The inequality $C_{EUR} \le S e^{-\delta T}$ follows from the fact that you would never pay more for a European call than you would for a prepaid forward expiring at time *T*. Since American options can be exercised immediately, we can set $T = 0$ in the previous inequalities to obtain $S - K \le C_{AM} \le S$.
4. The inequality $K e^{-rT} - S e^{-\delta T} \le P_{EUR}$ follows from put-call parity, along with the observation that $C_{EUR} \ge 0$. The inequality $P_{EUR} \le K e^{-rT}$ follows from the fact that you would never pay more for a European put than you would for a bond that paid *K* at time *T*. Since American options can be exercised immediately, we can set $T = 0$ in the previous inequalities to obtain $K - S \le P_{AM} \le K$.

Arbitrage

If any of the inequalities above are violated, then opportunities for arbitrage exist. The details of constructing an arbitrage will depend on the situation, but the general idea is to buy an underpriced asset and sell an overpriced asset.

Example 1.8

The current price of a stock is 120. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate of interest is 4%. A two-year 150-strike put on the stock with a price of 21 is currently available.

An arbitrage opportunity exists by buying or selling the put, buying or selling *K* shares of the stock, and by borrowing or lending *B* dollars. Find *K* and *B* and determine what assets need to be bought and sold, and if the arbitrage involves borrowing or lending.

Time Until Expiration

Assume that $T_1 < T_2$. Since an American option expiring at time T_2 is able to be exercised at time T_1 , we have:

- $C_{AM}(S, K, T_1) \le C_{AM}(S, K, T_2)$
- $P_{AM}(S, K, T_1) \le P_{AM}(S, K, T_2)$

The previous statements can be summarized by stating that the value of an American option is an increasing function of *T*, the time until expiration. The longer until expiration, the greater the value of the American option.

Generally speaking, less can be said for about the relative prices of European options with different settlement dates, especially if the underlying asset pays dividends. The following two statements will hold true if the asset does NOT pay dividends.

- $C_{EUR}(S, K, T_1) \le C_{EUR}(S, K, T_2)$. This follows from the fact that American and European calls are equivalent for non-dividend stocks. (See Section 2.2.)
- $C_{EUR}(S, K, T_1) \le C_{EUR}(S, K e^{-r(T_2 - T_1)}, T_2)$ and $P_{EUR}(S, K, T_1) \le P_{EUR}(S, K e^{-r(T_2 - T_1)}, T_2)$

Calls on Non-Dividend Assets

It can be shown that the value of an American call on a non-dividend paying stock at any moment is always greater than or equal to the payoff received for exercising the option at that moment. As a result, it is never optimal to exercise an American call on such a stock early. If the owner of the call wished to no longer own the call, it would be better to sell the call than exercise it.

Since early-exercise of an American call on a non-dividend stock is never optimal, such calls are actually equivalent to their European counterparts. This leads us to the following conclusion.

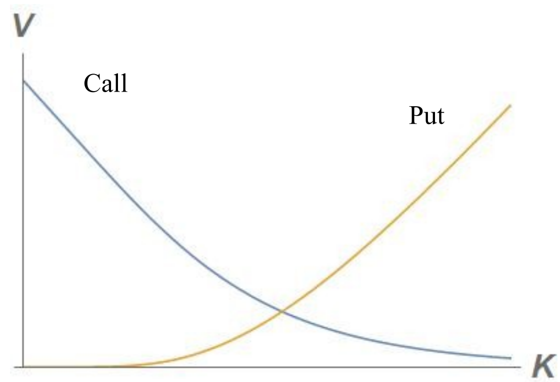
If the underlying asset pays no dividends, then $C_{EUR} = C_{AM}$.

1.5 DIFFERENT STRIKE PRICES: MONOTONICITY

Monotonicity

The owner of a call must pay the strike price if the call is exercised. If two otherwise equivalent calls have different strike prices, the one with the higher strike will be less desirable, and will thus have a lower premium. This line of reasoning illustrates that a call premium must be a decreasing function of *K*. Since the owner of a put receives the strike price if the put is exercised, similar logic leads us to conclude that put premiums are increasing functions of *K*.

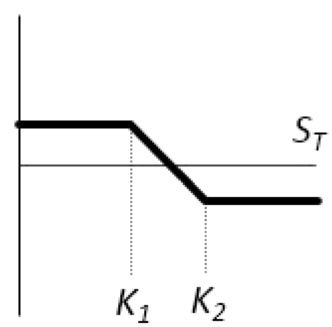
Typical plots for call and put premiums as functions of *K* are shown in figure below.



Difference in Premiums for Calls with Difference Strikes

Consider two *T*-year European calls on the same stock, one with strike K_1 and one with strike K_2, where $K_1 < K_2$. Denote the call premiums by $C_{EUR}(K_1)$ and $C_{EUR}(K_2)$. We have shown that $C_{EUR}(K_1) > C_{EUR}(K_2)$, and thus $C_{EUR}(K_1) - C_{EUR}(K_2) > 0$. We now wish to establish an upper bound for $C_{EUR}(K_1) - C_{EUR}(K_2)$.

To establish the desired result, we construct a $K_1 - K_2$ bear spread by buying the K_2-strike call and selling the K_1-strike call. The cost of the spread is $C_{EUR}(K_2) - C_{EUR}(K_1)$. Note that for a bear spread, the lowest profit at expiration occurs when $S_T > K_2$. In that case, the payoff of the spread is equal to $K_1 - K_2$, and the overall profit is equal to $K_1 - K_2 - FV\left[C_{EUR}(K_2) - C_{EUR}(K_1)\right]$. This value must be negative. If not, we would have a spread that is always profitable, which would provide an opportunity for arbitrage. Noting that the profit must be negative when $S_T > K_2$ allows us to establish the following string of inequalities:



- $0 > K_1 - K_2 - FV\left[C_{EUR}(K_2) - C_{EUR}(K_1)\right]$
- $FV\left[C_{EUR}(K_2) - C_{EUR}(K_1)\right] > K_1 - K_2$
- $C_{EUR}(K_2) - C_{EUR}(K_1) > PV(K_1 - K_2)$
- $C_{EUR}(K_1) - C_{EUR}(K_2) < PV(K_2 - K_1)$

A similar result holds for American calls with different strike prices. American options can, in theory, be exercised immediately, in which case $PV(K_2 - K_1) = K_2 - K_1$. This leaves us with $C_{AM}(K_1) - C_{AM}(K_2) < K_2 - K_1$.

The results in this subsection can be summarized by noting that the difference in the price of two call options with different strikes must be less that the difference in the strike prices. In the case of European options, we have the stronger result that the difference in call prices is less than the present value of the difference in the strike prices.

Example 1.9

Prices for three 2-year European calls with strike prices 50, 52, and 55 are given as follows: $C(50) = 5.97$, $C(52) = 3.80$, and $C(55) = 1.31$. An opportunity for arbitrage exists. Construct a spread that could take advantage of this opportunity using no more than one of each option. Find the potential range of profits for this spread. Assume a risk-free rate of 4%.

Difference in Premiums for Puts with Difference Strikes

Consider two *T*-year European put on the same stock, one with strike K_1 and one with strike K_2, where $K_1 < K_2$. Using an argument very similar to the one we employed with calls (although using a bull spread rather than a bear spread), we can show that if the puts are European, then $P_{EUR}(K_2) - P_{EUR}(K_1) < PV(K_2 - K_1)$ and if the puts are American, then $P_{AM}(K_2) - P_{AM}(K_1) < K_2 - K_1$. Compare these inequalities to the ones obtained for calls, noting the slight differences. These differences are attributable to calls being decreasing functions of *K* while puts are increasing functions of *K*.

Example 1.10

Prices for American calls and puts with strike prices 100, 104, and 110 are provided below. Determine if any opportunities for arbitrage exist. If so, state what options need to be purchased to exploit the opportunity or opportunities.

- $C(100) = 14$, $C(104) = 11$, $C(110) = 9$
- $P(100) = 7$, $P(104) = 9$, $P(110) = 16$

Bounds on Slope

Since calls are decreasing functions of *K*, the slope of the graph of *C* as a function of *K* is always negative. Stated in terms of partial derivatives, we have that $C_K = \delta C / \delta K \leq 0$. It is also possible to establish a lower bound for C_K. Whether working with either American or European calls, if $K_1 < K_2$ then $C(K_1) - C(K_2) < K_2 - K_1$. This inequality can be rewritten as $\left[C(K_2) - C(K_1)\right] / \left(K_2 - K_1\right) > -1$. This states that the average rate of change in *C* if the strike changes from K_1 to K_2 is greater than -1. Since this is true for all K_1 and K_2 where $K_1 < K_2$, the same result holds for the instantaneous rate of change, and so $C_K > -1$. Thus, we see that $-1 < C_K < 0$.

Using similar logic, we may show that for both American or European puts, the slope of the graph of *P* as a function of *K* is always less than 1. Since *P* is an increasing function of *K*, we have that $0 < P_K < 1$.

Summary of Monotonicity Results

The results that we have established relating to monotoniticy and slope in this section are summarized below.

Calls	**Puts**
For both American and European calls: • *C* is a decreasing function of *K*. • $-1 < C_K < 0$ If $K_1 < K_2$, then: • $C_{AM}(K_1) - C_{AM}(K_2) < K_2 - K_1$ • $C_{EUR}(K_1) - C_{EUR}(K_2) < PV(K_2 - K_1)$	For both American and European puts: • *P* is an increasing function of *K*. • $0 < P_K < 1$ If $K_1 < K_2$, then: • $P_{AM}(K_2) - P_{AM}(K_1) < K_2 - K_1$ • $P_{EUR}(K_2) - P_{EUR}(K_1) < PV(K_2 - K_1)$

1.6 DIFFERENT STRIKE PRICES: CONVEXITY

Convexity

Let V be the value of any standard option. It can be shown that $V_{KK} > 0$, and thus the graph of V as a function of K is concave up, or convex. The upward concavity of V allows us to say that if $K_1 < K_2 < K_3$, then $V(K_2)$ must be less than the linearly interpolated estimate obtained by using $V(K_1)$ and $V(K_3)$. More precisely:

- Since K_2 lies between K_1 and K_2, it can be written as a linear combination of the other two values, with the coefficients summing to 1.
- Let $a = \dfrac{K_3 - K_2}{K_3 - K_1}$ and $b = \dfrac{K_2 - K_1}{K_3 - K_1}$. Then $K_2 = a\,K_1 + b\,K_3$.
- The linearly interpolated (over)estimate of $V(K_2)$ would then be $a\,V(K_1) + b\,V(K_3)$.
- Convexity assures us that $V(K_2) < a\,V(K_1) + b\,V(K_3)$.

Example 1.11

The prices for American calls and puts on a stock are provided below for three difference strike prices. Determine if any opportunities for arbitrage exist with the options. For each such opportunity, construct a spread to take advantage of the opportunity.

Strike	85	100	125
Call	16	10	2
Put	4	12	20

Example 1.12

The prices for American calls a stock are provided below for three difference strike prices. Find the range of values for X that would NOT result in arbitrage opportunities.

Strike	50	X	60
Call	7	3	2

Example 1.13

Let $C(K)$ and $P(K)$ denote the premiums for a T-year, K-strike European call and put (respectively) on one share of ABC stock, which is currently worth S. Determine which, if any, of the following statements are true.

a) $0 \le C(100) - C(105) \le 5e^{-rT}$

b) $0 \le P(100) - P(105) \le 5e^{-rT}$

c) $100e^{-rT} \le P(100) - C(105) + S \le 105e^{-rT}$

d) $C(100) - 3C(105) + 2C(115) > 0$

1.7 EARLY EXERCISE OF AMERICAN OPTIONS

Early Exercise in General

When you exercise an American option early, you lose the option (which you could have sold on the market), and you receive the payoff from exercising the option. Thus, you would only ever exercise an option early if the payoff from doing so was greater than the current value of the option. More specifically:

- **Calls:** You would exercise an American call at time *t* if and only if $C_{AM}(S_t, K, T-t) \le S_t - K$.
- **Puts:** You would exercise an American put at time *t* if and only if $P_{AM}(S_t, K, T-t) \le K - S_t$.

Example 1.14

One share of XYZ stock is currently worth 67. The prices for six-month European puts on a stock with strike prices 50, 55, 60, and 65 are given in the table below.

Strike Price	50	55	60	65
Put	0.0679	0.3609	1.2276	3.0123

Six months ago, Kyle purchased four special one-year call options on XYZ stock, one for each strike price listed above. The terms of the options allow Kyle the right to exercise the calls early, but only at six months. Being at the six-month mark, Kyle has to decide which options to exercise today, and which to hold for another six months (or sell). For which of the calls would it be optimal for Kyle to exercise today?

Calls on a Non-Dividend Paying Stock

It can be shown that if the underlying stock pays no dividends, then $C_{AM}(S_t, K, T-t) \ge S_t - K$ for all values of *t*. Thus, early exercise of an American call on a **non-dividend** paying stock is never optimal. It follows that such an option is equivalent to its European counterpart, and so $C_{EUR} = C_{AM}$ if the stock does not pay dividends.

Puts on a Non-Dividend Paying Stock

In contrast with the situation for calls, there may be times when early exercise of a put on a non-dividend paying stock is optimal. As mentioned above, an American option should be exercised at time t if and only if $P_{AM}(S_t, K, T-t) \le K - S_t$. Using inequalities from Section 2.1, as well as put-call parity, we can also show that early exercise of an American put is NOT optimal if $C_{EUR}(S_t, K, T-t) \ge K - PV(K)$. It is important to note that if this last inequality fails, that tells us nothing about early exercise of the put.

Options on Dividend Paying Stocks

We can use principles already introduced to find conditions under which we can be sure that early exercise is NOT optimal. These statements are not equivalent to early exercise being optimal. They can identify certain circumstances where early exercise is NOT optimal, but if they fail, we are told nothing about early exercise.

- Early exercise of an American call on a dividend paying stock is NOT optimal if $PV(\text{Div}) \le K - PV(K)$.
- Early exercise of an American put on a dividend paying stock is NOT optimal if $K - PV(K) \le PV(\text{Div})$.

1.8 SYNTHETIC STOCKS AND TREASURIES

Synthetic Stocks

Assume that a stock pays continuous dividends. Then $C(K,T) - P(K,T) = S_0 e^{-\delta T} - K e^{-rT}$, or $S_0 = e^{\delta T} C(K,T) - e^{\delta T} P(K,T) + K e^{(\delta - r)T}$. Thus, buying one share of the stock today is equivalent to buying $e^{\delta T}$ call options, selling $e^{\delta T}$ put options, and lending $K e^{(\delta - r)T}$ (by buying a zero-coupon bond).

Synthetic Treasuries

If a stock pays continuous dividends, then we may also write the equation for put-call parity as $K e^{-rT} = S_0 e^{-\delta T} - C(K,T) + P(K,T)$. Thus, lending $K e^{-rT}$ (by purchasing a zero-coupon bond), is equivalent to buying $e^{-\delta T}$ shares of the stock, selling one call, and buying one put. In either case, the portfolio will be worth *K* at time *T*.