# CHAPTER 2 – Binomial Trees

## 2.1 INTRODUCTION TO BINOMIAL TREES

The payoff for a put or a call is a function of the price of the underlying stock on the expiration date of the option. In order to appropriately price options, we first need construct a probabilistic model for the price of the stock. We can then use this model to determine the probability that a particular option will be exercised, as well as the expected payoff of the option. We can then use this information to price the option.

There are two commonly used stock price models: the binomial tree mode and the lognormal model. In this chapter, we will student the binomial tree model. The basic "one-period binomial tree" model that we start with is a very simplistic model, but we will see later that it serves as the building block for more complicated and more realistic stock models.

### One-Period Binomial Trees

A one-step binomial tree model is described as follows.

- Let *S* be the current price of the stock. This will sometimes be denoted by $S_0$ .
- The model covers a specific length of time. The period length is denoted by *h*, and is measured in years.
- We assume that there are only two possible values of the stock at time *h*. Either the price of the stock will increase to a value of $S_u$ , or it will decrease to a value of $S_d$ .
- The values $S_u$ and $S_d$ are sometimes stated explicitly, but are often provided as multiples of *S*. If the multipliers *u* and *d* are provided, then $S_u = S \cdot u$ and $S_d = S \cdot d$ .
- The probability of an up-move is denoted by $p$ . The probability of a down-move is equal to $q = 1 - p$ .

### Expected Stock Price and Expected Annual Return

Given a one-period binomial tree, we may calculate the following.

- The **expected price** of the stock after 1 period is $E\left[S_h\right] = p \cdot S \cdot u + q \cdot S \cdot d$ .
- The **capital gains rate** $g$ is the continuously compounded annual rate of growth that would cause the initial stock price of *S* to grow to the expected value $E\left[S_h\right]$. In other words, $S\,e^{gh} = E\left[S_h\right]$. It follows that $g = \frac{1}{h} \cdot \ln\left(\frac{E\left[S_h\right]}{S}\right) = \frac{\ln\left(p \cdot u + q \cdot d\right)}{h}$ .
- The **continuously compounded expected annual rate of dividend growth** for the stock is denoted by $\delta$ .
- The **continuously compounded expected annual rate of return** $\alpha$ for the stock is equal to the capital gains rate plus the continuously compounded expected rate of dividend grown, $\delta$ . That is, $\alpha = g + \delta$ .

Notice that the expected rate of return $\alpha$ is the expected return on investment for someone purchasing the stock. They would expect return of $g$ due to price changes, as well as an additional return of $\delta$ due to dividend growth. This gives an total return of $\alpha = g + \delta$ . For a risk-averse investor to be interested in a particular stock, they would require $\alpha$ to be larger than the risk-free rate $r$ .

**Example 2.1**

The price of a stock is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The stock pays dividends continuously at a rate of 2% and is currently worth 50. The period for the tree is 8 months. The probability of an up-move is $p = 0.65$.

a) Determine the expected price of the stock after 8 months.

b) Find the continuously compounded expected annual return for the stock over the 8 month period.

**Example 2.2**

The price of a stock is modeled by a one-period binomial tree with $u = 1.15$ and $d = 0.9$. The stock pays continuous dividends at a rate of 3% and is currently worth 100. The period for the tree is 9 months. The continuously compounded expected annual return for the stock is $\alpha = 11\%$. Determine the probability of an up-move.

## Capital Gains Rate Versus Mean Annual Growth Rate

Let $R_h$ be the continuously compounded rate of growth in the stock price (ignoring dividends). The value $R_h$ has two possible values at time *h*, and is thus a random variable. In the case of an up-move, $R_h = \ln u$. If there is a down-move, then $R_h = \ln d$. Let $\mu_h = E[R_h]$. Then $\mu_h = p \cdot \ln u + q \cdot \ln d$. We now define the **mean annual growth rate**, $\mu$, by annualizing $R_h$. This gives us $\mu = (p \cdot \ln u + q \cdot \ln d)/h$. Compare this quantity with *g*:

- $g = \dfrac{\ln(p \cdot u + q \cdot d)}{h}$ and $\mu = \dfrac{p \cdot \ln u + q \cdot \ln d}{h}$

These formulas are slightly different. There is an important, but subtle, distinction between the capital gains rate and the mean annual growth rate. Read the following explanations carefully and make sure you understand the distinctions. We will generally be more interested in *g* than $\mu$, but it is important that you understand both.

- The mean annual grown rate, $\mu$, is the annualized expected rate of growth in the stock price.
- The capital gains rate, *g*, is the annualized growth rate that results in the expected price of the stock.

**Example 2.3**

The price of a non-dividend-paying stock currently worth 100 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 3 months. The probability of an up-move is $p = 0.5$.

a) Find the capital gains rate for the stock.

b) Find the mean annual growth rate for the stock price.

## Expected Payoff and Return for Options

Given a binomial tree model for a stock, we can use the model to determine the expected payoff as well as the expected return for a call or put on that stock expiring at the end of the period. The details are provided below.

- Consider a *K*-strike European call and a *K*-strike European put on a stock, both expiring at the end of one period. We will denote the payoff of the call at the up-node by $C_u$, and the payoff of the call at the down-node by $C_d$. Similarly, we will denote the payoff of the put at the up and down nodes by $P_u$ and $P_d$, respectively. Formulas for these quantities are given by:
  - $C_u = \max\left[0,\, S_u - K\right]$ and $C_d = \max\left[0,\, S_d - K\right]$
  - $P_u = \max\left[0,\, K - S_u\right]$ and $C_d = \max\left[0,\, K - S_d\right]$
- The expected payoffs of the options are given by $E\left[\text{Call } PO\right] = p\,C_u + q\,C_d$ and $E\left[\text{Put } PO\right] = p\,P_u + q\,P_d$.
- The continuously compounded expected annual return for a particular option is denoted by $\gamma$. It is defined by $\text{Premium} \cdot e^{\gamma h} = E\left[\text{Option } PO\right]$, or $\gamma = \dfrac{1}{h} \ln\left(\dfrac{E\left[\text{Option } PO\right]}{\text{Premium}}\right)$.

**Example 2.4**

The price of a non-dividend-paying stock currently worth 50 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 8 months. The probability of an up-move is $p = 0.65$. An 8-month 112-strike call has a premium of 4.42. The continuously compounded risk-free rate is 4%.

a) Determine the expected call payoff and the expected return for the call.

b) Determine the expected put payoff and the expected return for the put.

**Example 2.5**

The price of a non-dividend-paying stock currently worth 120 is modeled by a one-period binomial tree with $u = 1.1$ and $d = 0.9$. The period for the tree is 6 months. A 6-month 125-strike put has a premium of 6.1915 and an expected yield of 8.4911%. Find the expected return for the stock.

## 2.2 REPLICATING PORTFOLIOS

Given a binomial tree for a stock, it is not difficult to calculate the expected payoff for an option on that stock. The premium for the option should then be the present value of this expected payoff. The issue we face is that the rate that we should use to discount the expected payoff is $\gamma$, the expected yield of the option. If we don't already know the premium, we don't have a method of calculating $\gamma$. This requires us to develop alternate methods for pricing options. We will learn two methods: replicating portfolios and risk-neutral pricing. We will cover replicating portfolios in this section, and risk-neutral pricing in the next.

### Replicating Portfolios

The method of replicating portfolios allows us to price options on a stock modeled by a binomial tree without using any probabilistic concepts. In this method, we will construct a portfolio consisting of some shares of the underlying asset, as well as some amount of borrowing of lending. The portfolio will be built so that the payoffs at time $h$ are exactly the same as the option in consideration at both the upper and lower nodes. Since those are the only two "possible" payoffs for the option in the binomial tree model, we conclude that the price of the option must be the same as the price of our replicating portfolio.

The process of pricing an option using a replicating portfolio is outlined below.

- Assume a stock is modeled using a 1 period binomial tree with $S$, $r$, $\delta$, $h$, $u$, and $d$ given.
- We construct a portfolio by buying $\Delta$ shares of the stock, and investing (lending) $B$ in risk-free bonds.
- The cost of our replicating portfolio is $\Delta S + B$.
- The portfolios value at $t = h$ is $\Delta e^{\delta h} S u + B e^{rh}$ at the up node and $\Delta e^{\delta h} S d + B e^{rh}$ at the down node.
- Let $V_u$ and $V_d$ be the payoffs for the option being priced at the upper and lower nodes respectively.
- For either a call or a put, we set $\Delta e^{\delta h} S u + B e^{rh} = V_u$ and $\Delta e^{\delta h} S d + B e^{rh} = V_d$. Solving this system yields $\Delta = \left( \dfrac{V_u - V_d}{Su - Sd} \right) e^{-\delta h}$ and $B = \left( \dfrac{u V_d - d V_u}{u - d} \right) e^{-rh}$.
- The price of the option is then $V = \Delta S + B$.
- Note that for a call, $\Delta \geq 0$ and $B \leq 0$. In contrast, for a put we have $\Delta \leq 0$ and $B \geq 0$.
- Let $\Delta_C$ be the number of shares in the replicating portfolio for a $K$-strike call and let $\Delta_P$ be the number of shares in the replicating portfolio for a $K$-strike put. A consequence of the previous derivations is that $\Delta_C - \Delta_P = e^{-\delta h}$.

We begin by looking at a few examples involving non-dividend-paying stocks.

**Example 2.6**

The price of a non-dividend-paying stock currently worth 130 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4.5%.

a) Calculate the premium of a one-year 128-strike call on the stock.

b) Calculate the premium of a one-year 128-strike put on the stock.

**Example 2.7**

The price of a non-dividend-paying stock currently worth 120 is modeled by a one-period binomial tree with $u = 1.3$ and $d = 0.85$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4%. Find the strike price of a one-year call option whose replicating portfolio contains 0.5926 shares of the stock.

We will now consider an example involving dividend-paying stocks.

**Example 2.8**

The price of a stock currently worth 100 is modeled by a one-period binomial tree with $u = 1.3$ and $d = 0.8$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 5%.

c) Calculate the premium of a one-year 98-strike call on the stock.

d) Calculate the premium of a one-year 98-strike put on the stock.

**Example 2.9**

The price of a stock is modeled using a one-period binomial tree with a period of six months. The difference between the price of the stock at the upper and lower nodes is 48. The difference between the payoffs of a six-month *K*-strike call at the upper and lower nodes is 31. The stock pays continuous dividends at a rate of 3%. Find the number of shares in the replicating portfolio for a six-month *K*-strike put on the stock.

## Delta-Hedging

A **market maker** is an entity that facilitates the buying and selling of stocks, as well as financial derivatives such as options. A market maker is generally not interested in making a profit by holding a position with respect to a stock. They instead make money by charging small commissions to any individual using their services to conduct a trade, whether that individual is buying an asset or selling an asset.

The ideal situation for a market maker is that whenever they sell an option, they are also able to buy an identical option. The market maker would collect commissions on both transactions and would maintain a neutral position with respect to the underlying option. When the number of options bought and sold are not equal, the market maker can make up the deficit by creating synthetic option using replicating portfolios. Replicating portfolios are constructed by buying or shorting shares of the stock and borrowing or lending some amount of money.

**Example 2.10**

The price of a certain stock follow a one-period binomial tree model with $u = 1.3$ and $d = 0.7$. The period for the tree is six months. The price of the stock is currently 80. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate is 5%.

A market-maker writes 100 six-month at-the-money put options on the stock. Determine the number of shares of the stock the market-maker must buy or sell to delta-hedge her portfolio.

We will discuss delta hedging in more detail in Section 4.2.

## 2.3 RISK NEUTRAL PRICING

The method of risk neutral pricing provides an alternative to replicating portfolios for pricing options on stocks whose prices are modeled using binomial trees. The two methods produce the same results, but each have their own advantages. For some applications, the value of delta found using the replicating portfolio method might be of interest in its own rate. An advantage of risk neutral pricing is that it tends to be easier to apply when working with a multi-period binomial tree.

### Risk Neutral Pricing Method

When using risk neutral pricing, we assume that we are in a "risk-neutral" world where every investment is expected to grow at the risk-free rate. The details of the method are explained below.

- Assume that the underlying stock is modeled by a one period binomial tree with parameters $S$, $u$, $d$, $\delta$, and $h$. Also assume that the risk-free rate $r$ is given.
- To use the risk neutral model, we do not need to know the value of $p$. Recall that if we did know $p$, then we could be able to calculate the expected price of the stock $E\left[S_h\right] = p \cdot S \cdot u + q \cdot S \cdot d$**.** We could subsequently calculate the capital gains rate $g$ using $S e^{gh} = E\left[S_h\right]$, and the expected yield $\alpha = g + \delta$.
- When using risk neutral pricing, we assume that the expected yield is equal to $r$ and then calculate the **risk neutral probability** $p^*$ consistent with this return. This amounts to solving for $p^*$ in the equation $p^* \cdot S \cdot u + \left(1 - p^*\right) \cdot S \cdot d = S e^{(r-\delta)h}$.
- The quantity $E^*\left[S_h\right] = p^* \cdot S \cdot u + \left(1 - p^*\right) \cdot S \cdot d$ is called the **risk neutral expected value of the stock**.
- Solving for $p^*$ gives the formula $p^* = \dfrac{e^{(r-\delta)h} - d}{u - d}$.
- Assume now that we wish to price an option that has values of $V_u$ and $V_d$ at the up and downs. The **risk neutral expected payoff** of the option is given by $E^*[PO] = p^* V_u + (1 - p^*) V_d$.
- The premium is then obtained by discounting the risk neutral expected payoff using the risk-free rate. That is: $\text{Premium} = \left[p^* V_u + (1 - p^*) V_d\right] e^{-rh}$.

**Risk Neutral Pricing**

- $p^* = \dfrac{e^{(r-\delta)h} - d}{u - d}$ or $p^* = \dfrac{S_0 e^{(r-\delta)h} - S_d}{S_u - S_d}$
- Risk Neutral Expected Payoff: $E^*[PO] = p^* V_u + (1 - p^*) V_d$
- $\text{Premium} = \left[p^* V_u + (1 - p^*) V_d\right] e^{-rh}$

**Example 2.11** The price of a stock currently worth 100 is modeled by a one-period binomial tree with $u = 1.3$ and $d = 0.8$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 5%.

a) Use risk neutral pricing to price a one-year 98-strike call on the stock.

b) Use risk neutral pricing to price a one-year 98-strike put on the stock.

The problem we just solved is identical to the one presented in Example 2.8. Compare the results to verify that the two methods do in fact generate the same prices.

**Example 2.12**

The price of a stock currently worth 140 is modeled by a one-period binomial tree with $u = 1.25$ and $d = 0.85$. The probability of an up-move is $p = 0.4$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 4.5%. Find the expected yield for a one-year 135-strike European put on the stock.

**Example 2.13**

The price of a nondividend-paying stock currently worth 40 is modeled by a one-period binomial tree with $u = 1.25$ and $d = 0.75$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4%. A 1-year, $K$-strike European put on the stock currently costs 4.8237. Find $K$.

**Example 2.14**

The price of a stock is modeled using a one-period binomial tree, with the period being one-year. The ratio of the true probability of an up-move to the risk-neutral probability of an up-move is 1.15. The continuously compounded risk-free rate of return is 5%. A one-year, $K$-strike call has a delta of $\Delta = 0.72$. Determine the continuously compounded expected rate of return for the call.

**Example 2.15**

The price of a nondividend-paying stock currently worth 80 is modeled by a one-period binomial tree with $u = 1.25$ and $d = 0.75$. The period for the tree is one year. A one-year, 90-strike European call on the stock currently costs 5.93. Determine the risk-free rate of interest.

## Exotic Options

Binomial trees can be used to price many exotic (i.e. nonstandard) options that we will discuss later in the course. In those cases, the strategy is similar: find the payoff of the option at each terminal node, use risk-neutral probabilities to find the (risk-neutral) expected payoff of the option at expiration, and then discount the expected payoff using the risk free rate to find the price of the option.

## 2.4 VOLATILITY

### Introduction to Volatility

When choosing a model for the price of a stock, an analyst would want the selected model to capture traits relating to the historical or expected future behavior of the stock. One obvious feature what would want to be captured would be the capital gains rate for the stock. The capital gains rate by itself does not give us enough information to effectively select a model, however. There can be many different binomial tree models with the same $g$. Consider the following one-period binomial tree models, for example.

a) $h = 1$, $S_0 = 100$, $u = 1.3$, $d = 0.9$, $p = 0.5$
b) $h = 1$, $S_0 = 100$, $u = 1.5$, $d = 0.7$, $p = 0.5$

In both cases, the stock being modeled has a capital gains rate of 10%. But, these models are quite different. If we were using these models to price an option, we would get very different answers. For example, assume we were pricing at at-the-money call and an at-the-money put for these stocks. Without actually calculating the priced, see if you can determine for which model the call is more valuable, and for which model the put is more valuable.

The primary difference between these two models is that prices at the in nodes in Model 2 are further apart, making the stock following that model more prone to wild swings in the price. In other words, prices for the stock in Model 2 are more volatile than prices for the stock in Model 1. We formally define the concept of volatility as the standard deviation in the continuously compounded return for the stock.

### Annual Volatility

Let $R$ be the continuously compounded rate of growth price of the stock over the course of the next year. Stated in another way, $S_1 = S_0 e^R$. We do not currently know the value of $R$, so it is a random variable. We define the **annual volatility, $\sigma$,** of the stock to be the standard deviation of $R$. In other words, $\sigma^2 = \text{Var}[R]$. Note that since $S_1 = S_0 e^R$, we have $R = \ln(S_1 / S_0)$ and $\sigma^2 = \text{Var}[\ln(S_1 / S_0)]$.

**Example 2.16**

Calculate the annual volatility for each of the following one-period binomial tree models.

a) $h = 1$, $S_0 = 100$, $u = 1.3$, $d = 0.9$, $p = 0.5$
b) $h = 1$, $S_0 = 100$, $u = 1.5$, $d = 0.7$, $p = 0.5$

## Periodic Volatility

The stock models we work with in this course sometimes cover a period of time less then one year. To determine the volatility of a stock following such a model, we introduce the concept of periodic volatility, and explain how it relates to annual volatility.

Consider an $h$-year period beginning at time 0. Let $R_h$ be the continuously compounded periodic return for a stock during this period. In other words, $S_h = S_0 e^{R_h}$. The **periodic volatility, $\sigma_h$,** of the stock is defined to be the standard deviation of $R_h$. In other words, $\sigma_h^2 = \mathrm{Var}\left[R_h\right]$, or $\sigma_h^2 = \mathrm{Var}\left[\ln\left(S_h / S_0\right)\right]$.

Let $n = 1/h$ be the number of $h$-year periods in one year. Let $R_1$, $R_2$, …, $R_n$ be the periodic returns for the stock during each of the $n$ periods. Let $R$ denote the annual return. Then $S_1 = S_0 e^{R_1} e^{R_2} \cdots e^{R_n} = S_0 e^{R_1 + R_2 + \ldots + R_n} = S_0 e^{R}$, and thus $R = R_1 + R_2 + \ldots + R_n$. We will assume that the periodic returns, $R_1$, $R_2$ …, $R_n$ all follow the same distribution, are independent of one-another, and all have a volatility of $\sigma_h$. Then:

$$\sigma^2 = \mathrm{Var}[R] = \mathrm{Var}\left[R_1 + R_2 + \ldots + R_n\right] = \mathrm{Var}\left[R_1\right] + \mathrm{Var}\left[R_2\right] + \ldots + \mathrm{Var}\left[R_n\right] = n\sigma_h^2 = \frac{1}{h}\sigma_h^2$$

It follows that the relationship between periodic and annual volatility is given by $\sigma = \sigma_h / \sqrt{h}$ or $\sigma = \sigma_h \sqrt{n}$.

**Example 2.17**

The price of a stock is modeled using a one-period, 3-month binomial tree with $u = 0.6$ and $d = 0.4$. The probability of an up-move is 60%. Determine the annual volatility of the stock.

The most important formulas relating to annual volatility and periodic volatility are summarized below.

**Annual and Periodic Volatility Formulas**

- $\sigma_h^2 = \mathrm{Var}\left[\ln\left(\dfrac{S_h}{S_0}\right)\right]$
- $\sigma^2 = \dfrac{1}{h}\mathrm{Var}\left[\ln\left(\dfrac{S_h}{S_0}\right)\right]$
- $\sigma = \dfrac{\sigma_h}{\sqrt{h}}$

## Modeling Volatility

Given a binomial tree model, we have seen how to calculate the volatility of the stock being modeled. We now turn our attention to determining how to create a model that has a certain desired volatility. In particular, given $\sigma$, we want to determine values of $u$, $d$, and $p$ that will result in a model with a volatility of $\sigma$. In fact, we will see that there are many possible values that could achieve this goal.

Consider a 1 period, *h*-year binomial tree. Assume, for convenience, that $p = 0.5$. Let $R_h$ be given by $S_h = S_h e^{R_h}$. Then $R_h = \ln\left(S_h / S_0\right)$ and either $R_h = \ln(u)$ or $R_h = \ln(d)$, each with a probability of 50%. This gives us $E\left[R_h\right] = \dfrac{\ln u + \ln d}{2}$ and $E\left[R_h^2\right] = \dfrac{(\ln u)^2 + (\ln d)^2}{2}$. We can substitute into $\sigma^2 = \dfrac{1}{h}\cdot\mathrm{Var}\left[R_h\right] = \dfrac{1}{h}\cdot\left[E\left[R_h^2\right] + E\left[R_h\right]^2\right]$ to obtain $\sigma^2 = \dfrac{1}{h}\left[\dfrac{(\ln u)^2 + (\ln d)^2}{2} - \left(\dfrac{\ln u + \ln d}{2}\right)^2\right] = \dfrac{1}{h}\left[\dfrac{(\ln u - \ln d)^2}{4}\right] = \dfrac{\ln(u/d)^2}{4h}$. Taking square roots of both sides yields $\sigma = \dfrac{\ln(u/d)}{2\sqrt{h}}$, and solving for $\dfrac{u}{d}$ gives us $\dfrac{u}{d} = e^{2\sigma\sqrt{h}}$.

This does not give give us specific values for *u* and *d*, but rather a condition that the two value have to satisfy. We will now provide several commonly used models that satisfy this condition.

**Binomial Tree Models**

- **Forward Tree (Standard/Usual Method)**
  $u = e^{(r-\delta)h + \sigma\sqrt{h}}$ and $d = e^{(r-\delta)h - \sigma\sqrt{h}}$
- **Lognormal Tree (Jarrow-Rudd Tree)**
  $u = e^{(r-\delta-0.5\sigma^2)h + \sigma\sqrt{h}}$ and $d = e^{(r-\delta-0.5\sigma^2)h - \sigma\sqrt{h}}$
- **Cox-Ross-Rubinstein Tree**
  $u = e^{\sigma\sqrt{h}}$ and $d = e^{-\sigma\sqrt{h}}$

The Forward Tree model is sometimes called the "Usual Method in McDonald". We will use this model much more frequently than the other two. Note that we assumed $p = 0.5$ in the discussion above. If *p* has a different value, then these models will not result in a volatility exactly equal to $\sigma$. However, if *p* is not far from 0.50, the resulting volatility should be close to $\sigma$.

**Example 2.18**

Construct a one-period binomial forward tree using the following parameters:

- $h = 1$
- $S_0 = 80$
- $\sigma = 30\%$
- $r = 5\%$
- $\delta = 3\%$

The continuously compounded expected return on the stock is 12%. Find the continuously compounded expected annual return on a one-year at-the-money put on the stock.

**Example 2.19**

The price of a stock is modeled by a one-period Cox-Ross-Rubinstein tree with a volatility of 25%. The period for the tree is three months. The stock currently has a price of 200 and pays dividends continuously at a rate of 3%. The risk-free rate is 5%.

A three-month, 210-strike call on the stock has a continuously compounded expected annual yield of 42%. Determine the continuously compounded expected return for the stock.

## Arbitrage

It can be shown that if a binomial tree model does not satisfy the inequality $d < e^{(r-\delta)h} < u$, then there will be opportunities for arbitrage. Of the three models we have considered, only the forward tree model guarantees that this no-arbitrage condition is met. Under "usual" circumstances, the Cox-Ross-Rubinstein and Lognormal trees will also satisfy the requirement, but there is no guarantee that these models will satisfy the criteria in general.

## Historical Volatility

A common, although not always valid, assumption in the stock models we consider in the course is that the returns for a stock during any two non-overlapping segments of time with equal length will follow the same distribution. For example, let $R_1$, $R_2$, …, $R_n$ be the periodic returns for a stock during $n$ consecutive $h$-year periods., and let $r_1$, $r_2$, …, $r_n$ denote the observed values of these returns. We will assume that each $R_i$ follows the same distribution. We can use the observed returns $r_i$ to calculate the sample standard deviation $\hat{\sigma}_h$ of the returns, which we will use as an estimate of the periodic volatility $\sigma_h$. We can then use the value $\hat{\sigma} = \hat{\sigma}_h / \sqrt{h}$ as an estimate for the annual volatility in the stock. This quantity is called the **historical volatility** of the stock since it is calculate using past data. A summary of the formulas relating to historical volatility are provided below.

**Historical Volatility**

- Let $S_0, S_1, S_2, \ldots, S_n$ be the prices of a stock observed at $h$-year intervals .
- Let $r_i = \ln\left(S_i / S_{i-1}\right)$ be the periodic return during the $i$th period.
- The historical volatility is given by $\hat{\sigma} = \dfrac{\hat{\sigma}_h}{\sqrt{h}}$, where $\hat{\sigma}_h = \sqrt{\dfrac{1}{n-1}\sum\left(r_i - \bar{r}\right)^2}$ .

We can use the TI-30XS to find the estimate $\hat{\sigma}_h$. The process is as follows:

- Press the **[data]** button and then enter the values of $r_i$ into the list **L1**.
- Press **[2nd]**, **[stat]**, and then select "**1-Var Stats**".
- Select "**L1**" for Data and "**ONE**" for **FRQ**.
- Then $\hat{\sigma}_h = Sx$ .
- Note that the value given for $\bar{x}$ is an estimate for the mean return $\mu_h$, and not the capital gains rate $g_h$ .

**Example 2.20**

The prices for a stock at the end of each of the first seven months of the year are given in the table below. Use these prices to estimate the annual volatility for the stock.

| Month | Jan | Feb | Mar | Apr | May | Jun | Jul |
|---|---|---|---|---|---|---|---|
| Stock Price | 80 | 88 | 92 | 82 | 85 | 78 | 84 |

**Example 2.21**

The prices for a stock at the end of each of the first six months of the year are given in the table below. The historical annual volatility based on these prices is 41.05%. Find $K$.

| Month | Jan | Feb | Mar | Apr | May | Jun |
|---|---|---|---|---|---|---|
| Stock Price | $K$ | 112 | 106 | 95 | 112 | $K$ |

## 2.5 MULTI-PERIOD BINOMIAL TREES

As mentioned previously, one-period binomial trees are not a particularly realistic model for stock prices. We can obtain more realistic models by expanding upon the idea and considering multi-period binomial trees. The details of the multi-period binomial tree model are explained below.

- Assume that the length of time covered by the model is $T$ years, and that this interval of time is split into $n$ periods of length $h$.
- The initial stock price is $S$. Prices at later nodes are denoted using subscripts indicating the number of up and down moves required to reach that node.
- The probability of an up-move for any given period is $p$. In the case of an up-move, the price is multiplied by a factor of $u$. The multiplier for a down-move is $d$.

### Pricing European Options Using Multi-Period Binomial Trees

We will explain how to price a European Option using a two-period binomial tree. The process for using a binomial tree with more than two periods is a natural extension to this method.

1. Assume that we are pricing a $2h$-year $K$-strike European option.
2. Denote the payoffs of the option at each of the three terminal nodes by $V_{uu}$, $V_{ud}$, and $V_{dd}$.
3. Use the payoffs $V_{uu}$ and $V_{ud}$ to calculate the price of a $h$-year $K$-strike option of the same type, sold at time $h$, assuming that an up-move occurred during the first period. Denote this quantity by $V_u$.
4. Use the payoffs $V_{ud}$ and $V_{dd}$ to calculate the price of a $h$-year $K$-strike option of the same type, sold at time $h$, assuming that a down-move occurred during the first period. Denote this quantity by $V_d$.
5. Use the values $V_u$ and $V_d$ to determine the price of the option, $V$.

An alternate (and equivalent) method would be to calculate the risk-neutral expected payoff at time $2h$ using $E^*\left[S_{2h}\right] = \left(p^*\right)^2 S_{uu} + 2\,p^* q^* S_{ud} + \left(q^*\right)^2 S_{dd}$, and then discount to time 0 using the risk-free rate: $V = E^*\left[S_{2h}\right]e^{-2rh}$.

**Example 2.22**

The price of a stock currently worth 160 is modeled by a two-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is 6 months. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 6%.

a) Calculate the premium for a one-year 185-strike European call on the stock.

b) Calculate the premium for a one-year 185-strike European put on the stock.

**Example 2.23**

The price of a dividend-paying stock currently worth 100 is modeled using a 3-period binomial tree with $u = 1.1$ and $d = 0.9$. Each period is one year. The continuously compounded risk-free rate of interest is 8%. The price of a three-year 120-strike European call on the stock is 3.4117. Find the price of a three-year 100-strike European call on the stock.

**Example 2.24**

The price of a nondividend-paying stock currently worth 100 is modeled using a 12-period binomial tree using the usual method in McDonald. The volatility of the stock is 35%. The continuously compounded risk-free rate is 6%. Find the premium for a one-year 135-strike European call on the stock.

## Pricing American Options Using Multi-Period Binomial Trees

Recall that American options and European options differ in that European options can only be exercised when the option expires, whereas an American option can be exercised at any point prior to the expiration date for the option. We can use multi-period binomial trees to price American options by making a small adjustment to the process using for European options. We explain the process for a two period binomial tree below.

1. Assume that we are pricing a $2h$-year $K$-strike American option. We also assume that the option in question can only be exercised at the end of an $h$-year period.
2. Denote the payoffs of the option at each of the three terminal nodes by $V_{uu}$, $V_{ud}$, and $V_{dd}$.
3. We now calculate $V_u$. The process is more complicated than with European options.
   a) Use the payoffs $V_{uu}$ and $V_{ud}$ to find the price of a $h$-year $K$-strike option of the same type, sold at time $h$, assuming that an up-move occurred during the first period. Denote this quantity by $MV_u$.
   b) Find the payoff for the option at the up-node if it were exercised early. Denote this by $PO_u$.
   c) Let $V_u = \max\left[MV_u, PO_u\right]$.
4. We now calculate $V_u$.
   a) Use the payoffs $V_{ud}$ and $V_{dd}$ to find the price of a $h$-year $K$-strike option of the same type, sold at time $h$, assuming that a down-move occurred during the first period. Denote this quantity by $MV_d$.
   b) Find the payoff for the option at the down-node if it were exercised early. Denote this by $PO_d$.
   c) Let $V_d = \max\left[MV_d, PO_d\right]$.
5. Use the values $V_u$ and $V_d$ to determine the price of the option, $V$.

**Example 2.25**

The price of a stock currently worth 160 is modeled by a two-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is 6 months. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 6%.

a) Calculate the premium for a one-year 185-strike American call on the stock.

b) Calculate the premium for a one-year 185-strike American put on the stock.

c) Find the value of delta for a one-year 185-strike American call on the stock.

d) Find the value of delta for a one-year 185-strike American put on the stock.

**Example 2.26**

The price of a nondividend-paying stock currently worth 100 is modeled by a four-period binomial tree with $u = 1.1$ and $d = 0.9$. Each period is one year. The continuously compounded risk-free rate is 4%. A four-year 90-strike American call on the stock is priced using this binomial tree model.

The price of the stock decreases during each of the first two years. Find the value of delta for the the call at the end of the second year.

**Example 2.27**

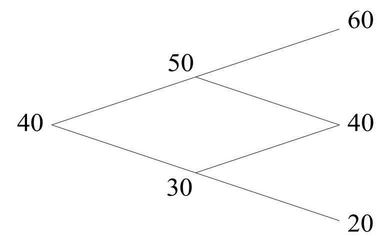
The price of a paying stock currently worth 125 and paying continuous dividends at a rate of 2% is modeled by a two-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is six months. The continuously compounded risk-free rate is 8%. You purchase a one-year $K$-strike put on the stock. During the first six months, the price of the stock declines. Determine the smallest value of $K$ for which early exercise would be optimal at the end of the first six months.

Although it is not common to do so, it is possible to construct a multi-period binomial tree in which each the one-period subtrees have different values for $u$ and $d$. Options on a stock modeled by such a tree can be priced using standard techniques, although a new risk-neutral probability will have to be calculated for each subtree.

**Example 2.28**

The price of a stock is modeled using the two-period binomial tree on the right. Each period is six months. The stock pays continuous dividends at a rate of 2%. The risk-free rate of interest is 8%.

Calculate the premium for a one-year 50-strike European put on the stock.



## 2.6 UTILITY

### Introduction to Utility

- Let $r$ and $\delta$ be the risk-free rate and the dividend rate, both stated as annual **effective** rates.
- Let $W_u$ and $W_d$ represent the "worth" of an additional dollar at the up node and at the down node, respectively.
- Define $W_H = \frac{p^*}{p}$ and $W_L = \frac{1-p^*}{1-p}$.
- Then $pW_H + (1-p)W_L = 1$.
- Then $p\left(\frac{1}{1+r}W_H\right) + (1-p)\left(\frac{1}{1+r}W_L\right) = \frac{1}{1+r}$.
- Let $U_H = \frac{1}{1+r}W_H$ and $U_L = \frac{1}{1+r}W_L$. Then $pU_H + (1-p)U_L = \frac{1}{1+r}$.
- Let $Q_H = pU_H$ and $Q_L = (1-p)U_L$. Then $Q_H + Q_L = \frac{1}{1+r}$.

### Pricing using utility:

- $Q_H = \frac{p^*}{1+r}$ and $Q_L = \frac{1-p^*}{1+r}$.
- $S = [Q_H S_u + Q_L S_d](1+\delta)$
- $V = Q_H V_u + Q_L V_d$