# CHAPTER 3 – The Black-Scholes Framework

## 3.1 THE LOGNORMAL DISTRIBUTION

Let $R_t$ be the continuously compounded periodic return for a stock during from time 0 to time *t*. In other words, $S_t = S_0 e^{R_t}$. Notice that at time 0, neither $R_t$ or $S_t$ have been observed, and are thus both random variables. These variables are obviously closely linked in that if we know the value of one, we will know the value of the other. They will almost certainly follow different probability distributions, however.

Under the lognormal stock model, we will assume that the return $R_t$ follows a normal distribution. As a consequence, $S_t$ follows a distribution known as the lognormal distribution. Before discussing this stock model in detail, we should cover some basic facts relating to the normal and lognormal distribution.

### Standard Normal CDF

Let $Z \sim \text{Normal}(0,1)$ be the standard normal distribution. The cumulative distribution function (CDF) for the normal distribution is given by $N(z) = \text{Prob}[Z \le z]$. It should be noted that $\Phi(z)$ is traditionally used to denote the standard normal CDF, and our use of $N(z)$ is somewhat nonstandard.

Let $X \sim \text{Normal}(m, v^2)$ be a normally distributed variable with mean $m$ and standard deviation $v^2$. Values for the CDF of X can be calculated using the standard normal CDF as follows: $\text{Prob}[X \le x] = N\left(\dfrac{x-m}{v}\right)$.

An equation that is often useful when working with the standard normal distribution is $N(-z) = 1 - N(x)$.

We will also make frequent use of the inverse standard normal CDF, $N^{-1}(p)$. This function is defined as follows: $N^{-1}(p) = z$ if and only if $N(z) = p$.

### Definition of Lognormal Distribution

Let $X \sim \text{Normal}(m, v^2)$ be a normally distributed random variable with mean $m$ and standard deviation $v$. Let $Y = e^X$. Then we say that $Y$ has a **lognormal distribution** defined by the parameters $m$ and $v$. Several properties of the lognormal distribution are stated below without proof. The pdf of the distribution is also shown.

**Properties of Lognormal Distribution**

- $E[Y] = e^{m + 0.5v^2}$
- $\text{Var}[Y] = \left(E[Y]\right)^2\left[e^{v^2} - 1\right] = e^{2m + v^2}\left(e^{v^2} - 1\right)$
- $\text{Med}[Y] = e^m$
- $\text{Mode}[Y] = e^{m - v^2}$

## Lognormal Probability Calculations

Let $X \sim \text{Normal}(m, v^2)$ and $Y = e^X$. We can calculate $\text{Prob}[Y \le k]$ as follows:

$$\text{Prob}[Y \le k] = \text{Prob}[e^X \le k] = \text{Prob}[X \le \ln k] = \text{Prob}\left[Z \le \frac{\ln k - m}{v}\right] = N\left(\frac{\ln k - m}{v}\right)$$

**Example 3.1**

Let $X$ be normally distributed with a mean of 3 and a standard deviation of 2 and let $Y = e^X$.

a) Find $P[Y \le 100]$.

b) Find the 10th, 50th, and 90th percentiles of $X$.

c) Find the 10th, 50th, and 90th percentiles of $Y$.

**Example 3.2**

Assume that $X$ is normally distributed with mean 0.2 and standard deviation 0.3 and let $Y = 50e^X$. Find $P[Y \le 35]$.

## Constant Multiples of a Lognormal Random Variable

Let $X \sim \text{Normal}(m, v^2)$ be a normally distributed random variable with mean $m$ and standard deviation $v$. Let $Y = e^X$. Then, by definition, $Y$ is a lognormal random variable. Now, let $Z = CY = Ce^X$, where C is a constant. Then $Z = Ce^X = e^{\ln C}e^X = e^{\ln C + X}$. Let $W = \ln C + X$. Since $X$ is normally distributed with mean $m$ and standard deviation $v$, $W$ is also normally distributed, with mean $m + \ln C$ and standard deviation $v$. Since $Z = e^W$, we can conclude that $Z$ is lognormally distributed.

In summary, if $Y$ is a lognormal random variable, then $Z = CY$ will also follow a lognormal distribution, though with different parameters from $Y$.

## 3.2 THE LOGNORMAL STOCK MODEL

In this section, we see how to construct a stock model using the lognormal distribution. Our models will be of the form $S_t = S_0 e^{R_t}$, with the *t*-year return $R_t$ normally distributed. Let *m* and *v* be the mean and standard deviation for $R_t$. The question we need to answer is the following: To what should the parameters *m* and *v* be set so as to produce certain desired properties of the stock model? In particular, we would like to be able to specific the capital gains rate and the volatility of the stock being modeled.

### Lognormal Parameters

Let $S_0$ be the current price of a stock for which we would like to construct a model. Assume that the stock pays continuous dividends at a rate of $\delta$ and is expected to have a capital gains rate of *g* and a volatility of $\sigma$. We wish to use the lognormal distribution to make predictions about value of $S_t$. To that end, we will set $S_t = S_0 e^{R_t}$ and assume that $R_t$ is normally distributed. We will now derive expressions for the parameters *m* and *v* for $R_t$.

Let $S_t = S_0 e^{R_t}$ where $R_t \sim \text{Normal}(m, v^2)$. Then $\frac{S_t}{S_0} \sim LogN(m,\ v^2)$.

- **Finding *v*.** Recall from Section 2.4 that $\sigma^2 = \frac{1}{t}\text{Var}\left[\ln\left(\frac{S_t}{S_0}\right)\right]$. It follows that $\sigma^2 = \frac{1}{t}\text{Var}[R_t] = \frac{1}{t}v^2$. This tells us that $v^2 = \sigma^2 t$ and $v = \sigma\sqrt{t}$.
- **Finding *m*.** Recall from Section 2.1 that $g = \frac{1}{t}\cdot\ln\left(E\left[\frac{S_t}{S_0}\right]\right)$. Using the formula provided in Section 3.1 for the expected value of a lognormally distributed variable, we obtain $g = \frac{1}{t}\ln\left(e^{m + 0.5v^2}\right) = \frac{1}{t}\left(m + 0.5v^2\right)$. Substituting in the previously obtained expression for *v* and solving for *m*, we get $m = \left(g - 0.5\sigma^2\right)t$. Since $g = \alpha - \delta$, *m* can be written as $m = \left(\alpha - \delta - 0.5\sigma^2\right)t$.

### Relationships Between Different Growth Rates

The values $g$, $\delta$, $\alpha$, $m$, and $\mu$ are closely related and easy to confuse. The following discussion aims to clarify the difference and the relationships between these quantities.

- $g$ is the **capital gains rate**. It satisfies the expression $S_0 e^{gt} = E[S_t]$.
- $\delta$ is the continuously compounded **dividend rate** for the stock.
- $\alpha$ is the **expected annual yield**. It includes growth due to price changes and dividends: $\alpha = g + \delta$
- $\mu$ is the **mean annual return**, defined as $\mu = E[R]$. For the lognormal model, we have $\mu = \alpha - \delta - 0.5\sigma^2$.
- $m$ is the **mean *t*-year** return, defined by $m = E[R_t]$. We know that $m = \left(\alpha - \delta - 0.5\sigma^2\right)t$, or $m = \mu t$.

The formulas for the parameters *m* and *v* are summarized below.

**Summary of Lognormal Parameters**

Let $S_t = S_0 e^{Rt}$ where $Rt \sim \text{Normal}(m, v^2)$. Then:

- $m = \left(\alpha - \delta - 0.5\sigma^2\right)t$, $m = \left(g - 0.5\sigma^2\right)t$, and $m = \mu t$ where $\mu = \alpha - \delta - 0.5\sigma^2$
- $v = \sigma\sqrt{t}$

**Example 3.3**

The current price for one share of a certain stock is 80. The stock pays dividends continuously at a rate of 3% and has an expected yield of 11%. The volatility of the stock is 20%. The risk-free rate is 5%. Assume the stock follows the lognormal model.

a) Find the expected price of the stock after one year..

b) Find the median price of the stock after one year.

c) Find the probability that the price of the stock after 1 year is greater than 110.

d) Find the probability that the price of the stock after 2 years is greater than 110.

e) Find the 10th percentile of the price of the stock after 1 year.

f) Find the 10th percentile of the price of the stock after 2 years.

## Methods of Stating Volatility

Exam problems will state the volatility of a stock in many indirect ways. Some common methods are given below.

**Methods of Stating Volatility**

- $\text{Var}\left[\ln\left(S_T / S_0\right)\right] = \sigma^2 T$
- $\text{Var}\left[\ln\left(S_T\right)\right] = \sigma^2 T$
- $\text{Var}\left[\ln\left(F_{0,T}\right)\right] = \text{Var}\left[\ln\left(F^P_{0,T}\right)\right] = \sigma^2 T$
- $\ln\left[\dfrac{E\left[S_T\right]}{\text{Med}\left[S_T\right]}\right] = 0.5\sigma^2 T$

## Confidence and Prediction Intervals

Assume that $R_t \sim \text{Normal}\left(m, v^2\right)$ and $S_t = S_0 e^{A t}$. Let $0 \le p \le 1$ and $Z_{p/2} = N\left(1 - p/2\right)$.

- The $100\left(1 - p\right)\%$ –confidence interval for $R_t$ is given by $\left(m - z_{p/2} v,\ \ m + z_{p/2} v\right)$.
- The $100\left(1 - p\right)\%$ –prediction interval for $S_t$ is given by $\left(S_0 e^{m - z_{p/2} v},\ \ S_0 e^{m + z_{p/2} v}\right)$.

It should be noted that the confidence interval for $R_t$ is symmetric about the $\text{Med}\left[R_t\right] = m$, but the prediction interval for $S_t$ is NOT symmetric about $\text{Med}\left[S_t\right] = S_0 e^m$.

**Example 3.3**

Assume the prices for a stock are modeled follow a lognormal model. You are given that $E\left[S_4\right] = 160$ and $\ln\left[\dfrac{E\left[S_4\right]}{\text{Med}\left[S_4\right]}\right] = 0.18$. Find the 90% prediction interval for $S_4$.

**Example 3.4**

The current price for a stock is 50. The stock pays continuous dividends at a rate of 3% and has an expected annual yield of 15%. The volatility of the stock is 40%. Assume the stock follows the lognormal model. The $100\beta\%$ -prediction interval for $S_2$ is $\left(26.2343,\ K\right)$. Find $K$ and $\beta$.

**Example 3.5**

The price of a stock follows a lognormal model. The stock is currently worth 100, pays dividends at a rate of 2%, and a volatility of 30%. The 95%-prediction interval for the price of the stock after three years is $\left(K,\ 307.54\right)$. Find the expected annual yield for the stock.

## 3.3 ESTIMATING EXPECTED YIELD AND VOLATILITY

We saw in Section 2.4 that we can estimate a stock's volatility calculating the sample standard deviation of the yield rates for the stock over several consecutive periods. We called this estimate "historical volatility". In this section, we will take a similar approach to estimate not only the stock's volatility $\sigma$, but the stock's expected annual yield $\alpha$. The process is described below.

- Let $S_0, S_1, S_2, \ldots, S_n$ be the prices of a stock observed at $h$-year intervals over a period of $n \cdot h$ years.
- Let $r_i = \ln\left(S_i / S_{i-1}\right)$ be the observed periodic return during the $i$th period.
- We will assume that the periodic returns are independent and identically distributed. In other words, we assume that the $r_i$ 's are independent observations of a random variable $R_h$.
- Let $\hat{m}$ and $\hat{v}$ be the sample mean and sample standard deviation of the $r_i$ values. So, $\hat{m} = \left(\sum r_i\right) / n$ and $\hat{v} = \left[\sum \left(r_i - \hat{m}\right)^2\right] / \left(n - 1\right)$.
- We use formulas from Section 3.2 to obtain estimates for $\alpha$ and $\sigma$. These estimates are given by $\hat{\sigma} = \frac{\hat{v}}{\sqrt{h}}$ and $\hat{\alpha} = \frac{\hat{m}}{h} + \delta + 0.5\hat{\sigma}^2$.

We summarize these results below.

**Estimating Expected Yield and Volatility**

- Let $S_0, S_1, S_2, \ldots, S_n$ be the prices of a stock observed at $h$-year intervals.
- Calculate $r_i = \ln\left(S_i / S_{i-1}\right)$. Then find $\hat{m} = \frac{1}{n}\sum r_i$ and $\hat{v} = \frac{1}{n-1}\sum\left(r_i - \hat{m}\right)^2$.
- Our estimates are given by $\hat{\sigma} = \frac{\hat{v}}{\sqrt{h}}$ and $\hat{\alpha} = \frac{\hat{m}}{h} + \delta + 0.5\hat{\sigma}^2$.

As mentioned in Section 2.4, we can use the **1-Var Stats** calculator function to quickly find $\hat{m}$ and $\hat{v}$.

**Example 3.6**

A stock pays dividends continuously at a rate of 3%. Assume the stock follows a lognormal model. Prices for the stock on several different dates are provided below.

| Date | Jan 1, 15 | Mar 1, 15 | May 1, 15 | Jul 1, 15 | Sep 1, 15 | Nov 1, 15 | Jan 1, 16 |
|---|---|---|---|---|---|---|---|
| Price | 63.58 | 71.82 | 65.19 | 56.37 | 68.63 | 77.94 | 66.57 |

a) Estimate the expected annual return and the volatility for the stock.

b) Find the expected price of the stock at the end of 2016.

**Example 3.7**

A stock pays dividends continuously at a rate of 2%. Assume the stock follows a lognormal model. Prices for the stock on several different dates are provided below.

| Date | Jan 1, 15 | Apr 1, 15 | Jul 1, 15 | Oct 1, 15 | Jan 1, 16 |
|---|---|---|---|---|---|
| Price | $K$ | 117 | 131 | 154 | 142 |

Based on this information, the historical volatility of the stock is 25.15%. Estimate the expected annual return for the stock.

**Example 3.8**

A stock pays dividends continuously at a rate of 1%. Assume the stock follows a lognormal model. Prices for the stock on several different dates are provided below.

| Date | Jan 1, 15 | Feb 1, 15 | Mar 1, 15 | Apr 1, 15 | May 1, 15 | Jun 1, 15 | Jul 1, 15 |
|---|---|---|---|---|---|---|---|
| Price | 52 | 61 | $X$ | $Y$ | $X$ | 59 | 54 |

Based on this information, the historical volatility of the stock is 39.68%. Estimate the expected annual return for the stock.

**Example 3.9**

A stock pays dividends continuously at a rate of 3%. Assume the stock follows a lognormal model. Prices for the stock on several different dates are provided below.

| Date | Jul 1, 15 | Aug 1, 15 | Sep 1, 15 | Oct 1, 15 | Nov 1, 15 | Dec 1, 15 | Jan 1, 16 |
|---|---|---|---|---|---|---|---|
| Price | $X$ | 86 | 92 | 81 | 74 | 84 | $X$ |

Based on this information, the expected annual return is estimated to be 9.48%. Estimate the volatility of the stock.

## 3.4 EXPECTED OPTION PAYOFF

In order to price options on a stock, we need to be able to calculate the expected payoff of the option. To perform that task, we much first be able to calculate the probability that the option is exercised, as well as the price of the stock given that the option is exercised. These objectives will be the subject of this section.

### Probability of Exercise

Consider a $t$-year, $K$-strike European call option on a stock. The call will only be exercised if $S_t > K$. The probability that the call will be exercised is thus equal to $Pr[S_t > K]$. Notice that:

$$Pr[S_t > K] = Pr\left[S_0 e^{R_t} > K\right] = Pr\left[e^{R_t} > K / S_0\right] = Pr\left[R_t > \ln(K / S_0)\right]$$

Assuming the lognormal model, the random variable $R_t$ is normally distributed with mean $m = (\alpha - \delta - 0.5\sigma^2)t$ and standard deviation $v = \sigma\sqrt{t}$. Continuing the derivation above, we get:

$$Pr[S_t > K] = Pr\left[Z > \frac{\ln(K / S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right] = 1 - N\left(\frac{\ln(K / S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)$$

We now use the fact that $1 - N(x) = N(-x)$ along with a little algebraic manipulation to obtain:

$$Pr[S_t > K] = N\left(-\frac{\ln(K / S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right) = N\left(\frac{\ln(S_0 / K) + (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)$$

For simplicity of notation, let $\hat{d}_2$ denote the quantity inside the standard normal CDF in the final expression. Thus, we have shown that $Pr[S_t > K] = N(\hat{d}_2)$. A very similar argument shows that $Pr[S_t < K] = N(-\hat{d}_2)$.

**Probability of Exercise of European Options**

- Let $\hat{d}_2 = \dfrac{\ln(S_0 / K) + (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}$.
- The probability of a $t$-year, $K$-strike European call being exercised is $Pr[S_t > K] = N(\hat{d}_2)$.
- The probability of a $t$-year, $K$-strike European put being exercised is $Pr[S_t < K] = N(-\hat{d}_2)$.

**Example 3.10**

Assume the prices for a stock follow the lognormal model. The current price of the stock is 120. The stock pays continuous dividends at a rate of 2%, has an expected annual return of 16%, and has a volatility of 40%. The continuously compounded risk-free rate is 4%. Find the probability that the following options on this stock will be exercised.

a) A two-year, 150-strike European call.

b) A two-year, 150-strike European put.

c) A four-year, at-the-money European call.

d) A four-year, at-the-money European put.

**Example 3.11** Assume that the prices for a stock follow a lognormal model. You are given the following:

- $S_0 = 70$
- $\delta = 3\%$
- $\text{Var}\left[\ln\left(F_{0,3}^P\right)\right] = 0.12$

The probability that a three-year, 80-strike call on the stock will be exercised is 80%. Find the continuously compounded expected return on the stock.

## Partial Expectation

Let $X$ be a continuous random variable with density function given by $f(x)$. The **partial expectation of *X*, given that $X < K$**, is denoted by $PE[X \mid X < K]$, and defined to be $PE[X \mid X < K] = \int_{-\infty}^{K} x\, f(x)\, dx$. This partial expectation can be thought of as the portion of the expected value that the values of $X$ less than $K$ contribute to. We will be interested in partial expectations for their use in the calculation of conditional expectations

We will now provide formulas for the partial expectations of lognormal random variables. The derivation of these formulas will be omitted. Let $X \sim \text{Normal}(m, v^2)$ and $Y = e^X$. Then:

$$PE[Y \mid Y < K] = E[Y]N\left(-\frac{m + v^2 - \ln K}{v}\right) \text{ and } PE[Y \mid Y > K] = E[Y]N\left(\frac{m + v^2 - \ln K}{v}\right)$$

To apply these formulas to the stock model yields the following:

$$PE[S_t \mid S_t < K] = S_0 e^{(\alpha - \delta)t} N\left(-\frac{\ln(S_0 / K) + (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)$$

$$PE[S_t \mid S_t > K] = S_0 e^{(\alpha - \delta)t} N\left(\frac{\ln(S_0 / K) + (\hat{\alpha} - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}\right)$$

To simplify the notation, we will let $\hat{d}_1 = \dfrac{\ln(S_0 / K) + (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}$. This gives us:

$$PE[S_t \mid S_t < K] = S_0 e^{(\alpha - \delta)t} N(-\hat{d}_1) \text{ and } PE[S_t \mid S_t > K] = S_0 e^{(\alpha - \delta)t} N(\hat{d}_1)$$

Note that $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{t}$.

**Partial Expectation**

- Let $\hat{d}_1 = \dfrac{\ln(S_0 / K) + (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}$ and $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{t}$.
- $PE[S_t \mid S_t < K] = S_0 e^{(\alpha - \delta)t} N(-\hat{d}_1)$ and $PE[S_t \mid S_t > K] = S_0 e^{(\alpha - \delta)t} N(\hat{d}_1)$

## Conditional Expectation

We now turn our attention to developing formulas for calculating the expected value of a stock *given* that the stock price is either greater or less than a certain threshold. In other words, we want to calculate the conditional expectation of a stock, given that a certain type of option was exercised.

In general, the conditional expectations $E[X \mid X < K]$ and $E[X \mid X > K]$ of a random variable are calculate using the formulas:

$$E[X \mid X < K] = \frac{PE[X \mid X < K]}{Pr[X < K]} \quad \text{and} \quad E[X \mid X > K] = \frac{PE[X \mid X > K]}{Pr[X > K]} .$$

Combining results from earlier in the section, we obtain the following formulas for conditional expectations for prices of stocks following the lognormal model:

**Conditional Expectation**

- Let $\hat{d}_1 = \dfrac{\ln(S_0 / K) + (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}$ and $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{t}$.
- $E[S_t \mid S_t < K] = \dfrac{S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)}{N(-\hat{d}_2)}$ and $E[S_t \mid S_t > K] = \dfrac{S_0 e^{(\alpha-\delta)t} N(\hat{d}_1)}{N(\hat{d}_2)}$

**Example 3.12**

Assume that the prices for a stock follow a lognormal model. You are given the following:

- $S_0 = 90$
- $\sigma = 30\%$
- $\alpha = 13\%$
- $\delta = 3\%$

a) Find $E[S_4 \mid S_4 < 100]$.

b) Find $E[S_4 \mid S_4 > 100]$.

c) Find $E[S_4]$.

**Example 3.13**

Assume prices for a stock follow the lognormal model. You are given the following:

- $\ln\left[E[S_3] / \text{Med}[S_3]\right] = 0.125$
- There is a 75.3063% chance that the price of the stock three years from now will be higher than the current price of the stock.
- If the stock price is higher three years from now, then the expected price will be 105.55.

Find the current price of the stock.

**Example 3.14**

The current price of a nondividend-paying stock is $S_0$. The continuously compounded expected annual yield for the stock is 8%. The continuously compounded risk-free rate of interest is 3%. According to the lognormal model, the probability that the time-*T* price of the stock is greater than $0.8S_0$ is 0.6586. The expected price of the stock at time *T*, given that the price is greater than $0.8S_0$, is $1.3128F_{0,T}$. Find the volatility of the stock.

## Expected Option Payoff

We now have all of the necessary tools for calculating the expected payoff of European options. For any given European option, the expected payoff will be equal to the probability that the call is exercised times the expected payoff of the option given that it is exercised. We will start by considering the situation for calls.

Consider a $t$-year, $K$-strike European call on a stock whose prices follow the lognormal model. We can calculated the expected payoff of the call as follows:

$$
\begin{aligned}
E[\text{Call PO}] &= \left(Pr[\text{Call is exercised}]\right)\left(E[PO \text{ of call} \mid \text{Call is exercised}]\right) \\
&= \left(Pr[S_t > K]\right)\left(E[S_t - K \mid S_t > K]\right) \\
&= N(\hat{d}_2)\left(E[S_t \mid S_t > K] - K\right) \\
&= N(\hat{d}_2)\left(\frac{S_0 e^{(\alpha-\delta)t} N(\hat{d}_1)}{N(\hat{d}_2)} - K\right) \\
&= S_0 e^{(\alpha-\delta)t} N(\hat{d}_1) - K\, N(\hat{d}_2)
\end{aligned}
$$

A similar argument can be used to show that $E[\text{Put PO}] = K\,N(-\hat{d}_2) - S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$. In summary:

**Expected Option Payoff**

- $E[\text{Call PO}] = S_0 e^{(\alpha-\delta)t} N(\hat{d}_1) - K\,N(\hat{d}_2)$ and $E[\text{Put PO}] = K\,N(-\hat{d}_2) - S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$

**Example 3.15**

Let $S_t$ be the time-*t* price of a stock paying dividends at a continuously compounded rate of 2%. The current price of the stock is 50, and $\ln(S_2 / S_0) \sim \text{Normal}(m = 0.08,\ v^2 = 0.32)$. Find the expected payoff of an two-year 60-strike European call on this stock.

**Example 3.16**

Assume prices for a nondividend-paying stock follow the lognormal model. The stock is currently worth 80 and an expected annual yield of 6%. You are also given that $\ln\left[E[S_4] / \text{Med}[S_4]\right] = 0.32$. Given that $\hat{d}_2 = 0.44607$, find the expected payoff of a four-year European put on this stock.

**Example 3.17**

You are given the following information about the time *T* price of a stock:

- The partial expectation of the price, given that it is less that than *K*, is 31.1647.
- The conditional expectation of the price, given that it is less than *K*, is 54.2328.
- $\text{Var}[\ln(S_T)] = 0.48$.

Find the expected payoff of a *T*-year *K*-strike put on the stock.

Since we do not know the yield rates on the options we are not able to discount the expected payoffs to time 0 in order to find the option premiums. This is similar to the issue we encountered with binomial trees when using true probabilities. In the next section, we will learn how to apply a risk-neutral version of the above analysis to price these options. That method will result in the what is called the Black-Scholes formula.

## 3.5 BLACK-SCHOLES FORMULA FOR STOCK OPTIONS

### Derivation of Black-Scholes Formula

In the previous section we learned how to calculated the expected payoff of a European call or put on a stock whose price follows the lognormal stock model. If we knew $\gamma$, the expected yield of the option, we could use that rate to discount the expected payoff to time 0 to find the premium. We do not typically know $\gamma$ prior to knowing the option premium, however.

This situation should be familiar. We encountered the same difficulty when using binomial trees to price options. The solution in that case was to switch from true probabilities to risk-neutral probabilities. We will use the same approach here.

We begin by defining risk-neutral versions of $\hat{d}_1$ and $\hat{d}_2$, which we will call $d_1$ and $d_2$. This is done by simply replacing the yield rate $\alpha$ with the risk-free rate $r$. This yields the following expressions:

$$d_1 = \frac{\ln(S_0 / K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S_0 / K) + (r - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Recall from the previous section that the expected payoffs for a European call and put are given by:

$$E[\text{Call PO}] = S_0 e^{(\alpha - \delta)t} N(\hat{d}_1) - K N(\hat{d}_2) \quad \text{and} \quad E[\text{Put PO}] = K N(-\hat{d}_2) - S_0 e^{(\alpha - \delta)t} N(-\hat{d}_1)$$

Risk-neutral versions of these quantities are given by:

$$E^*[\text{Call PO}] = S_0 e^{(r - \delta)t} N(d_1) - K N(d_2) \quad \text{and} \quad E^*[\text{Put PO}] = K N(-d_2) - S_0 e^{(r - \delta)t} N(-d_1)$$

We discount these expressions to time 0 using the risk-free rate to find the option premiums:

$$\text{Call} = S_0 e^{-\delta t} N(d_1) - K e^{-rt} N(d_2) \quad \text{and} \quad \text{Put} = K e^{-rt} N(-d_2) - S_0 e^{-\delta t} N(-d_1)$$

These formulas are called the **Black-Scholes option pricing formulas**.

**Black-Scholes Formula for Stock Options**

- Let $d_1 = \dfrac{\ln(S_0/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = \dfrac{\ln(S_0/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$.
- The price of a European call on a stock is given by $\text{Call} = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)$.
- The price of a European put on a stock is given by $\text{Put} = K e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$.

The formulas above apply when the stock pays continuous (or no) dividends. If the stock pays discrete dividends, you can price the option using a general formula discussed later in Section 3.7. Also keep in mind that the Black-Scholes formula is used to price European options, and is not valid for American options.

**Example 3.18**

Assume the Black-Scholes framework. The current price of a stock is 90. The stock pays dividends continuously at a rate of 1% and has a volatility of 30%. The risk-free rate is 4%.

a) Calculate the premium for a two-year 90-strike call on the stock.

b) Calculate the premium for a two-year 90-strike put on the stock.

**Example 3.19**

Assume the Black-Scholes framework. The current price of a stock is 40. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate is 6%. You are given the following information regarding a six-month European put on the stock: $d_1 = -0.464756$ and $d_2 = -0.641533$. Find the price of this put.

## 3.6 BLACK-SCHOLES FORMULA FOR CURRENCY OPTIONS

Although the most common application for the Black-Scholes formula is pricing options on stocks, this tool can be used to price options on many different types of assets. In this section, we will see how to use the Black-Scholes formulas to price options on currency exchanges.

### Notation and Terminology for Currency Exchanges

Assume that we are interested in converting between two currencies, one domestic and one foreign, at some point in the future. Options may be used to provide insurance by placing caps on the exchange rate.

- For convenience, we will denote the domestic currency with $\$_d$ and the foreign currency with $\$_f$ .
- Assume $r_d$ is the risk-free rate for domestic currency and $r_f$ is the risk-free rate on the foreign currency.
- Assume one unit of the foreign currency currently costs $x_0$ units of the domestic currency. That is, $\$_f 1 = \$_d x_0$ .
- A *K*-strike call on the foreign currency gives the right to purchase $\$_f 1$ for a price of $\$_d K$ .
- A *K*-strike put on the foreign currency gives the right to sell $\$_f 1$ for a price of $\$_d K$ .
- The "denominating" currency for a currency exchange options represents to domestic currency, or in other words, the currency that the strike price and premium are paid in. For instance, a "Euro-denominated call on a dollar" is an option that allows for the purchase of a dollar, and will have the strike price and option premium paid in Euros.

### Pricing Options on Currency Exchanges

Options on currency exchanges can be priced using the standard Black-Scholes formula by making a few substitutions. The process is described below.

**Black-Scholes Formula for Currency Exchange Options**

- Make the following substitutions: $r = r_d$ , $\delta = r_f$ , and $S_0 = x_0$ .
- Let $\sigma$ be the volatility in the exchange rate.
- Then $d_1 = \dfrac{\ln\left(x_0 / K\right) + \left(r_d - r_f + 0.5\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$ .
- The price of a European call on the exchange is $Call = x_0 e^{-r_f T} N\left(d_1\right) - K e^{-r_d T} N\left(d_2\right)$ .
- The price of a European put on the exchange is $Put = K e^{-r_d T} N\left(-d_2\right) - x_0 e^{-r_f T} N\left(-d_1\right)$ .

**Example 3.20**

Assume the Black-Scholes framework. The current yen-to-dollar exchange rate is 110 yen per dollar. The annual volatility of the exchange is 30%. The continuously compounded risk-free rate for yen is 3% and the continuously compounded risk-free rate for dollars is 5%. Calculate the premium for a 9-month, 100-strike, yen-denominated European put on one dollar.

**Example 3.21**

A financial institution knows that it will need to purchase €200,000 two years from now in order to satisfy certain obligations. The company decides to hedge against rising exchange rates and purchases options allowing them to pay no more than $250,000 for the euros. Find the price of these options, in dollars, given the following information:

- The current exchange rate is €0.8 per $1.
- The continuously compounded risk-free rate for dollars is 4%.
- The continuously compounded risk-free rate for euros is 3%.
- The volatility of the exchange rate is 20%.

## Option Duality

Recall that any call can be viewed as a put by switching the underlying asset and the strike asset, and vice-versa. This concept is known as option duality, and was discussed in Section 1.3. It can be summarized using the following equation: $C(A=Y,\ K=X) = P(A=X,\ K=Y)$. Option duality is particularly useful when working with options on currency exchanges.

## 3.7 GENERAL BLACK-SCHOLES FORMULA

Before moving on to discussed other specific applications of the Black-Scholes formula, it will be useful to introduce a more general version of the formula than the one introduced in Section 3.5. This new formula can be used to price options on a wide variety of assets, such as stocks, bonds, and futures.

Assume an asset (not necessarily a stock) currently has a value of $S$. Consider a European option on the asset expiring at time $T$ with strike price $K$. We can use the formulas below to price such an option.

**General Black-Scholes Formula**

- Let $F^P(S)$ be the prepaid forward price of the asset.
- Let $F^P(K)$ be the prepaid forward price of a payment of $K$ at time $T$.
- Let $d_1 = \dfrac{\ln\left(F^P(S)/F^P(K)\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$ and $d_2 = \dfrac{\ln\left(F^P(S)/F^P(K)\right) - 0.5\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$.
- The price of a European call on the asset is given by $\text{Call} = F^P(S)N(d_1) - F^P(K)N(d_2)$.
- The price of a European put on the asset is given by $\text{Put} = F^P(K)N(-d_2) - F^P(S)N(-d_1)$.

The values of $F^P(S)$ and $F^P(K)$ will depend on the type of assets being used for the option. However, if the strike asset is a fixed amount of cash, then we will always set $F^P(K) = Ke^{-rT}$.

### Alternate Formula for $d_1$

Prepaid forward prices are always the PV of forward prices, and thus $F^P(S) = F(S)e^{-rT}$ and $F^P(K) = F(K)e^{-rT}$. It follows that $F^P(S)/F^P(K) = F(S)/F(K)$, and thus $d_1 = \dfrac{\ln\left(F(S)/F(K)\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$. If the strike asset is a fixed amount of cash, then $F(K) = K$ and $d_1 = \dfrac{\ln\left(F(S)/K\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$. This provides us with two equivalent formulas for $d_1$. Each will have their advantages in different situations.

One application of the general Black-Scholes formula is to price options on stocks paying discrete dividends. This is illustrated in the next example.

**Example 3.22** Assume the Black-Scholes framework applies. The current price of a stock is 100. The stock is scheduled to pay a dividend of 5 in nine months. The continuously compounded risk-free rate of interest is 8%, and $\text{Var}\left[\ln\left(F^P_{0,1}\right)\right] = 0.04$. Find the price of a one-year 95-strike European put on this stock.

## 3.8 BLACK-SCHOLES FORMULA FOR OPTIONS ON FUTURES

In this section, we will apply the general Black-Scholes formula to price options on futures contracts. Recall that a futures contract is a financial instrument that locks in the price of an asset that is to be delivered at some later date. The underlying asset for a future is often a stock, an index, or a physical commodity such as oil, gold, or wheat. To simplify the development of our pricing formulas, we will assume that the options considered are on assets that do not pay dividends. However, the formulas we obtain are valid even if the asset underlying the futures contract does pay dividends.

Consider a futures contract on a commodity, and an option on that futures contract. Assume that the option expires at time $T$ and the futures contract expires at time $T+S$. Let $P$ be the value of the commodity today. The price of the futures contract, to be paid at time $T+S$, is $F_{0,T+S} = Pe^{r(T+S)}$. If we wanted to pay for the commodity prior to delivery, say at time $T$ instead of at time $T+S$, the appropriate price would be $Pe^{r(T+S)}$ discounted back to time $T$ using the risk-free rate. This would give $Pe^{rT}$. Notice that the quantity $Pe^{rT}$ is also equal to the price for a futures contract that would expire at time $T$, when the option expires. This is the price that should be compared to the strike price, which is what will actually be paid for the contract at time $T$ if the option is exercised.

The discussion in the previous paragraph indicates that the actual date of expiration for the futures contract does not have any influence on the price of the option. Regardless of when the futures contract expires, we will always use the price of a futures contract that expires at the same time as the option we are pricing.

A summary of the formulas involved in pricing options on futures is provided below.

**Black-Scholes Formula for Options on Futures**

- Let $F$ be the price for a futures contract expiring at time $T$.
- Let $K$ be the strike price for a European option on the futures contract.
- Let $d_1 = \dfrac{\ln(F/K) + 0.5\sigma^2}{\sigma\sqrt{T}}$ and $d_2 = \dfrac{\ln(F/K) - 0.5\sigma^2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$.
- The price of a European call on the futures contract is $\text{Call} = F e^{-rT} N(d_1) - K e^{-rT} N(d_2)$.
- The price of a European put on the futures contract is $\text{Put} = K e^{-rT} N(-d_2) - F e^{-rT} N(-d_1)$.

**Example 3.23**

Assume the Black-Scholes framework applies. The one-year futures price for a certain commodity is currently 40. The price of a one-year at-the-money European call on the futures contract is 5.33893. The continuously risk-free rate of interest is 4%. Find the price of a 35-strike one-year European call on the futures contract.

**Example 3.24**

Assume the Black-Scholes framework applies. The one-year futures price for a certain commodity is currently 70. The price of a one-year at-the-money European put on the futures contract is 7.86042. The continuously risk-free rate of interest is 6%. Three months later the nine-month futures price is 75. Find the value of the put at that time.

**Example 3.25**

Assume the Black-Scholes framework applies. You are given the following information: A four-year at-the-money European call on a futures contract for Commodity A currently costs 15 more than a four-year at-the-money European call on a futures contract for Commodity B. The volatility of futures prices for both commodities is 20%. Find the absolute difference in the prepaid forward prices of the two commodities.

## 3.9 BLACK-SCHOLES FORMULA FOR BOND OPTIONS

We will now look at how to price an option on a zero-coupon bond. We will apply the general Black-Scholes formula to develop our pricing formulas, but we first need to introduce some notation regarding the prices of zero-coupon bonds.

### Bond Price Notation

The notation introduced below is convenient for working with zero-coupon bonds. We will use this notation frequently in Chapter 8 when we study interest rate models.

- $P(t, T)$ denotes the price, set at time *t*, of a bond to be purchased at time *t*, maturing for \$1 at time *T*.
- $P_0(t, T)$ denotes the price, set at time 0, of a bond to be purchased at time *t*, maturing for \$1 at time *T*.
- Notice that $P_0(t, T)$ is the forward price of $P(t, T)$. As such, we also denote $P_0(t, T)$ by $F_{0,T}(t, T)$.
- Interest theory concepts tell us that $P_0(t, T) = \dfrac{P(0,T)}{P(0,t)}$.
- Notice that if risk free rate is constant, $P(0,T) = e^{-rT}$. Even if the risk free rate varies, we can use $P(0,T)$ as a present value factor. If we want the present value of a payment occurring at time *T*, we multiply by $P(0,T)$.

### Pricing Options on Bonds

The formulas used in pricing an option on a bond are provided below.

**Black-Scholes Formula for Options on Bonds**

- Consider a zero-coupon bond maturing for \$1 at time $T + S$.
- An option allows for the purchase or sale of the bond at time *T* for a strike price of *K*.
- The time *T* forward price of the bond is $F = F_{0,T}(t, T) = P_0(T,\ T + S) = \dfrac{P(0,\ T + S)}{P(0,\ T)}$.
- Let $d_1 = \dfrac{\ln(F / K) + 0.5\sigma^2}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.
- The price of a European call on the bond is given by $C = P(0,\ T)\left[F\,N(d_1) - K\,N(d_2)\right]$.
- The price of a European put on the bond is given by $P = P(0,\ T)\left[K\,N(-d_2) - F\,N(-d_1)\right]$.

## Additional Exercises

### Section 3.1

1.

### Section 3.2

2. Assume that prices for a stock follow the lognormal model. The current price of the stock is 60. The stock pays dividends continuously at a rate of $\delta$. You are given that $E[S_4] = 86$ and $E[\ln(S_4)] = 4.32934$. Find the volatility of the stock.

3. The current price for a stock is 120. The stock pays continuous dividends at a rate of 4% and has an expected annual yield of 16%. The volatility of the stock is 35%. Assume the stock follows the lognormal model. There is a 90% chance that the price of the stock in 3 years will be greater than *K*. Find *K*.