# CHAPTER 4 – The Greeks

## 4.1 THE GREEKS

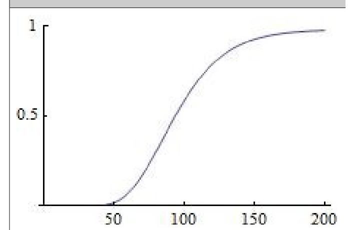
Let $V$ represent the value of an option that has been priced using the Black-Scholes formula. Then $V$ is a continuous function several variables, such as the stock price $S$, the strike price $K$, the asset volatility $\sigma$, the risk-free rate $r$, and the dividend rate $\delta$. We can differentiate $V$ with respect to any of these variables. **The Greeks** refer to a specific set of partial derivatives of $V$.

### Delta

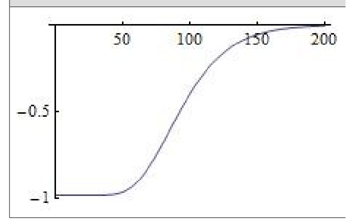
Delta, denoted by $\Delta$, is the partial derivative of $V$ with respect to the stock price $S$. That is to say that $\Delta = V_S$. Delta is perhaps the most widely used of the Greeks. As we will see later, it is closely related to the value of the same name that we studied for binomial trees. Several facts relating to delta are provided below.

- The value of a call increases as the stock price increases. However, the increase in the value of the call is always less than or equal to the increase in the stock price. Thus, $0 \le \Delta_C \le 1$.
- The value of a put decreases as the stock price increases. However, the decrease in the value of the put is always less than or equal to the increase in the stock price. Thus, $-1 \le \Delta_P \le 0$.
- It can be shown that, under the Black-Scholes framework, the delta for a call is given by $\Delta_C = e^{-\delta T} N(d_1)$ and the delta for a put is given by $\Delta_P = -e^{-\delta T} N(-d_1)$. The derivations of these formulas are omitted.
- Differentiating both sides of the put-call parity equation with respect to $S$ yields $\Delta_C - \Delta_P = e^{-\delta T}$.
- Graphs of $\Delta$ as a function of $S$ are shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Graph of $\Delta_C$




Graph of $\Delta_P$




**Example 4.1**

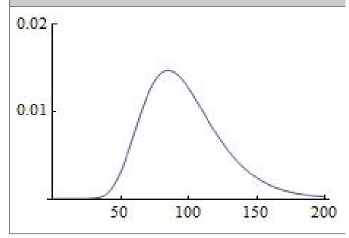
Assume the Black-Scholes framework applies. The continuously compounded risk-free rate of interest is 6%. A two-year at-the-money European call on the stock currently has a delta of 0.638265. An otherwise similar put has a delta of -0.322524. Determine the volatility of the stock.

## Gamma

Gamma, denoted by $\Gamma$, is the second partial derivative of $V$ with respect to the stock price $S$. That is to say that $\Gamma = V_{SS}$. Several facts relating to gamma are provided below.

- Gamma for a call is always equal to gamma for a put with the same parameters. That is, $\Gamma_C = \Gamma_P$.
- Gamma is always greater than or equal to zero.
- Gamma is given by the formula $\Gamma = \frac{1}{S\sigma\sqrt{2\pi T}} e^{-0.5\delta T d_1^2}$.
- A graph of $\Gamma$ as a function of $S$ is shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Graph of $\Gamma$

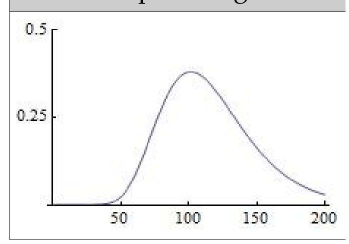



## Vega

Vega is the partial derivative of $V$ with respect to the volatility $\sigma$. That is to say that $vega = V_\sigma$. Several facts relating to vega are provided below.

- Vega for a call is always equal to vega for a put with the same parameters. That is, $vega_C = vega_P$.
- Vega is always greater than or equal to zero.
- Vega is occasionally defined as $vega = 0.01\,V_\sigma$. This definition of vega results from measuring $\sigma$ as a percentage.
- A graph of vega as a function of $S$ is shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Graph of vega

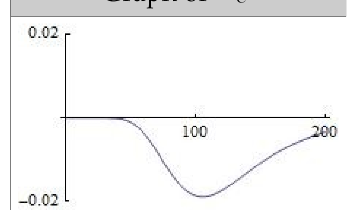



## Theta

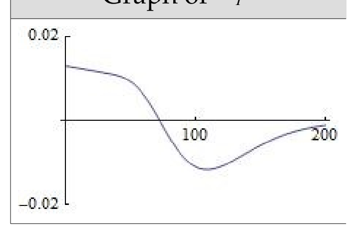
Theta, denoted by $\theta$, is the partial derivative of $V$ with respect to *t*, the time elapsed since the option was purchased. That is, $\theta = V_t$.

- By differentiating the expression for put-call parity, one can obtain the following relationship between theta for a call and theta for a put: $\theta_C - \theta_P = \delta S e^{-\delta(T-t)} - r K e^{-r(T-t)}$.
- Theta is occasionally defined to be $\theta = V_t / 365$. This definition of theta results from measuring *t* in days rather than years.
- Graphs of $\theta$ as a function of $S$ are shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Graph of $\theta_C$




Graph of $\theta_P$

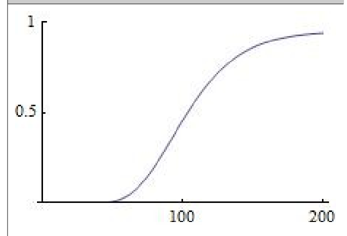


## Rho

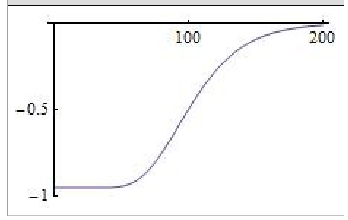
Rho, denoted by $\rho$, is the partial derivative of $V$ with respect to the risk-free rate *r*. That is, $\rho = V_r$.

- By differentiating the expression for put-call parity, one can obtain the following relationship between rho for a call and rho for a put: $\rho_C - \rho_P = T\,K\,e^{-rT}$.
- Rho for a call is always greater than or equal to zero. Rho for a put is always less than or equal to zero. These facts can be easily recalled by noting to role of *r* in the expressions $Call = x_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2)$ and $Put = K e^{-r_d T} N(-d_2) - x_0 e^{-r_f T} N(-d_1)$.
- Rho is occasionally defined to be $\rho = 0.01 V_r$. This definition of $\rho$ results from measuring $\sigma$ as a percentage.
- Graphs of $\rho$ as a function of *S* are shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Graph of $\rho_C$




Graph of $\rho_P$

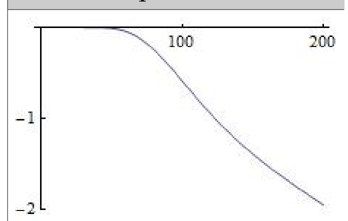



## Psi

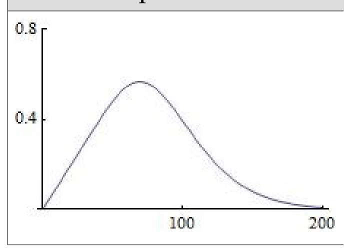
Psi, denoted by $\Psi$, is the partial derivative of $V$ with respect to the dividend rate $\delta$. That is, $\rho = V_\delta$.

- By differentiating the expression for put-call parity, one can obtain the following relationship between psi for a call and psi for a put: $\psi_C - \psi_P = -T\,S\,e^{-\delta T}$.
- Psi for a call is always less than or equal to zero. Psi for a put is always greater than or equal to zero. These facts can be easily recalled by noting to role of $\delta$ in the expressions $Call = x_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2)$ and $Put = K e^{-r_d T} N(-d_2) - x_0 e^{-r_f T} N(-d_1)$.
- Psi is occasionally defined to be $\Psi = 0.01 V_\delta$. This definition of $\Psi$ results from measuring $\sigma$ as a percentage.
- Graphs of $\Psi$ as a function of *S* are shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

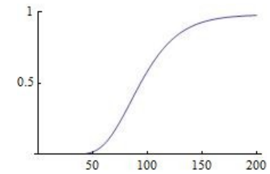
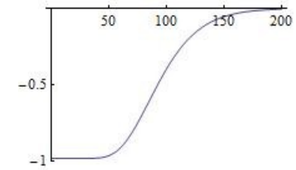
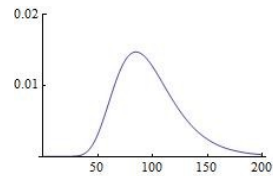
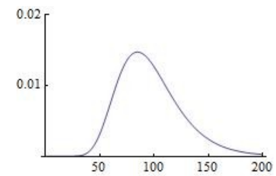
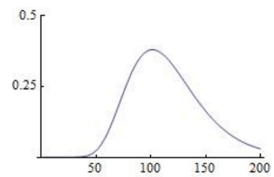
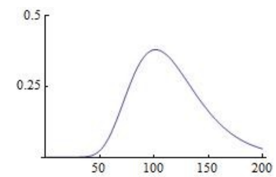
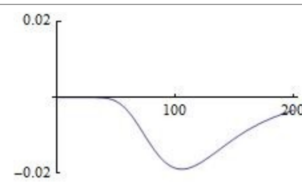
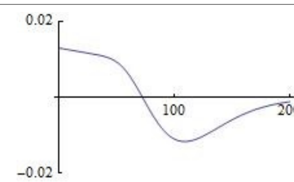
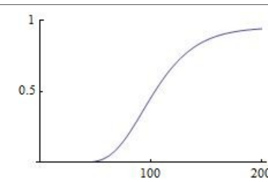
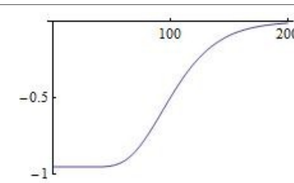
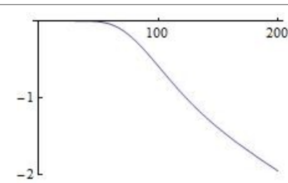
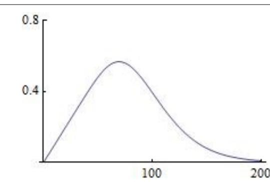
Graph of $\Psi_C$




Graph of $\Psi_P$



## Summary of the Greeks

| Greek | Derivative | Comments | Graph for Call | Graph for Put |
|---|---|---|---|---|
| Delta<br>$\Delta$ | $\frac{\partial V}{\partial S}$ | • $-1 \le \Delta_P \le 0 \le \Delta_C \le 1$<br>• $\Delta_C = e^{-\delta T} N(d_1)$<br>• $\Delta_P = -e^{-\delta T} N(-d_1)$<br>• $\Delta_C - \Delta_P = e^{-\delta T}$ |  |  |
| Gamma<br>$\Gamma$ | $\frac{\partial^2 V}{\partial S^2}$ | • $\Gamma_C = \Gamma_P$<br>• $\Gamma \ge 0$<br>• $\Gamma = \frac{1}{S\sigma\sqrt{2\pi T}} e^{-0.5\delta T d_1^2}$ |  |  |
| Vega | $\frac{\partial V}{\partial \sigma}$ | • $vega_C = vega_P$<br>• $vega \ge 0$<br>• Occasionally defined by $vega = 0.01\, V_\sigma$ |  |  |
| Theta<br>$\theta$ | $\frac{\partial V}{\partial t}$ | • The variable $t$ represents time that has elapsed since purchase of the option.<br>• $\theta_C - \theta_P = \delta S e^{-\delta(T-t)} - r K e^{-r(T-t)}$<br>• Occasionally defined by $\theta = V_t / 365$ |  |  |
| Rho<br>$\rho$ | $\frac{\partial V}{\partial r}$ | • $\rho_C - \rho_P = T K e^{-rT}$<br>• $\rho_C \ge 0$<br>• $\rho_P \le 0$<br>• Occasionally defined by $\rho = 0.01 V_r$. |  |  |
| Psi<br>$\Psi$ | $\frac{\partial V}{\partial \delta}$ | • $\psi_C - \psi_P = -T S e^{-\delta T}$<br>• $\psi_C \le 0$<br>• $\psi_P \ge 0$<br>• Occasionally defined by $\Psi = 0.01 V_\delta$. |  |  |

Note: The horizontal axis in each of the graphs above represents the stock price, $S$. The values of the other parameters are set at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

## Greeks for a Portfolio

Assume a portfolio consisting of $n$ options. Assume the current value of those options to the owner of the portfolio are given by $V_1, V_2, \ldots, V_n$. The the current value of the portfolio is given by $\pi = V_1 + V_2 + \ldots + V_n$. Let $G$ denote an arbitrary Greek. Assume that value of this Greek for the options in the portfolio are given by $G_1, G_2, \ldots, G_n$. The value of this Greek for the portfolio is given by:

- $$G_\pi = \frac{G_1 V_1 + G_2 V_2 + \ldots + G_n V_n}{V_1 + V_2 + \ldots + V_n} = \frac{G_1 V_1 + G_2 V_2 + \ldots + G_n V_n}{\pi}$$

In other words, the Greek for a portfolio is the weighted average of that Greek overall all of the assets in the portfolio, with the weights determined by the value of the assets.

## 4.2 DELTA HEDGING

### Market Makers

As discussed in Section 2.2, a **market maker** is an entity that facilitates the buying and selling of stocks, as well as financial derivatives such as options. A market maker is generally not interested in making a profit by holding a position with respect to a stock. They instead make money by charging small commissions to any individual using their services to conduct a trade, whether that individual is buying an asset or selling an asset.

The ideal situation for a market maker is that whenever they sell an option, they are also able to buy an identical option. The market maker would collect commissions on both transactions and would maintain a neutral position with respect to the underlying option. When the number of options bought and sold are not equal, the market maker can offset their position by buying or selling shares of the underlying stock. The most common strategy for using shares of a stock to offset an undesired position in that stock is the method of delta hedging.

### Delta Hedging

As described above, a market maker can buy or sell shares of a stock to offset a long or short position resulting from buying or selling an option on that stock. When employing the method of **delta hedging**, the number of shares that the market maker buys or sells will be equal to the delta of the option being offset.

If the market maker is attempting to offset a long position, they will sell shares of the stock. If the position being offset is short, then the market maker will purchase shares of the stock.

**Example 4.2**

Assume the Black-Scholes framework applies. The continuously compounded risk-free rate of interest is 5%. A share of ABC stock is currently worth 100. The stock pays continuous dividends at a rate of 2% and has a volatility of 30%. Determine the number of shares that must be bought or sold in order to delta hedge 1000 units of each of the following two-year European options on this stock.

a) A purchased 110-Strike Call.

b) A written 110-Strike Call.

c) A purchased 110-Strike Put.

d) A written 110-Strike Put.

To understand why delta hedging is an effective hedging strategy, assume that a portfolio consists of one option as well as $n$ shares of the underlying stock. Then the value of the portfolio will be given by $\pi = V + nS$. We would like the derivative of $\pi$ with respect to $S$ to be 0, so that small changes in the stock price will result in small changes in the value of the portfolio as a whole. Differentiating both sides of $\pi = V + nS$ with respect to $S$ yields $\pi_S = \Delta + n$. Setting $\pi_S = 0$ gives us $n = -\Delta$. This shows that the number of shares that we need to buy or sell is equal to $\Delta$. If $n$ turns out to be negative, then we should sell $S$ shares to offset the position. If $n$ is positive, then we should buy shares to offset the position.

## Delta Hedging a Portfolio

Assume we wish to delta hedge a portfolio consisting of several options, all with the same underlying stock. The options could be call or puts, purchased or written, and could have the same or different strike prices and times until expiration. To delta hedge this portfolio, we simply add in some number of stocks so that the overall delta for the portfolio is equal to zero. Pay careful attention to the signs of various quantities to determine whether the shares should be purchased or sold.

**Example 4.3**

Assume the Black-Scholes framework applies. The continuously compounded risk-free rate of interest is 4%. A share of nondividend-paying stock is currently worth 120. The volatility of the stock is 20%. A portfolio consists of the following European options:

- 100 one-year, 130-strike written calls
- 60 one-year, 130-strike written puts
- 80 two-year at-the-money purchased calls

Determine the number of shares that must be bought or sold in order to delta hedge this portfolio.

## Overnight Profit

We know the value of a delta hedged portfolio should change only slightly as a result of small changes in the stock price. It is likely, however, that the owner of such a portfolio will experience a small profit if the stock price changes. We will now discuss how to calculate the profit or loss for a delta hedged portfolio.

Assume that a portfolio of options on a certain stock currently has a value of $V_0$ and a delta of $\Delta$ .

- Denote the value of the unhedged portfolio at time *t* by $V_t$ .
- Denote the value of the hedged portfolio at time *t* by $\pi_t = V_t - \Delta e^{\delta t} S_t$ .
- The profit for the unhedged and hedged portfolios from *t* = 0 to *t* = *h* are given by
  - Unhedged: $Profit = V_h - FV(V_0)$ .
  - Hedged: $Profit = \pi_h - FV(\pi_0) = \left[V_h - \Delta e^{\delta h} S_h\right] - \left[V_0 - \Delta S_0\right] e^{rh}$
- It can be shown that the portfolio will have a positive profit if $S_0 - S_0 \sigma \sqrt{h} < S_h < S_0 + S_0 \sigma \sqrt{h}$ .

**Example 4.4**

Assume the Black-Scholes framework applies. A market maker sells 200 one-year, at-the-money European call options on a nondividend-paying stock and delta hedges the position. The current price for one share of the stock is 100, and the volatility of the stock is 40%. The continuously compounded risk free rate is 5%.

a) Find the number of shares bought or sold in the hedging portfolio.
b) Find the overnight profit for the hedged and unhedged portfolio if the price of the stock after one day is 95, 100, or 105.
c) Find the stock prices that would result in the market-maker breaking even after the one day.

**Example 4.5**

The current price of a nondividend-paying stock is 120. A market maker creates a portfolio by longing 60 one-year 125-strike European calls and shorting 40 one-year 130 strike European calls. The market maker delta hedges the positions. You are given the following information regarding the options:

- The 125-strike calls each have a premium of 13.74 and a delta of 0.5454.
- The 130-strike calls each have a premium of 11.80 and a delta of 0.4933.

One day later, the stock price has increased to 122.

- The new premium for the 125-strike calls is 14.83.
- The new premium for the 130-strike calls is 12.78.

Calculate the market maker's overnight profit. Assume $r = 3\%$.

**Example 4.6**

An investor buys 100 units of a $T$-year European put on a nondividend-paying stock and delta-hedges the position. At this time, the price of the stock is 70, the premium for the put is 4.66, and the put delta is -0.2311.The premium for an equivalent call option is 18.15.

At time $t < 1$, the investor closes out the portfolio. When she closes the portfolio, the stock price is 60 and the put premium is 4.31. The premium for an equivalent call option is 5.20.

Determine the investor's profit or loss.

## Rebalancing Portfolios

Note that $\Delta$ for a call or a put is a function of $S$, $K$, $T$, $\delta$, $r$, and $\sigma$. The strike price of an option is certainly fixed, and we tend to work under the assumption that $\delta$, $r$, and $\sigma$ will remain constant over the lifetime of the option as well. However, the stock price $S$ and the time until expiration $T$ will certainly change over time. As a result, $\Delta$ will change over time a delta hedged portfolio must continually be rehedged (or rebalanced) by buying or selling shares of the stock.

**Example 4.7**

A market maker sells 200 6-month European puts on a stock that pays dividends continuously at a rate of 3%. The market maker immediately delta hedges the position, and then rehedges after one week. The continuously compounded risk-free rate of interest is 4%. You are given:

- At the time of the sale the stock price is 150, the premium for one unit of the put is 14.15, and delta for the put is -0.4362.
- One week after the sale, the stock price is 160, the premium for one unit of the put is 10.03, and the delta for the put is -0.3371.

a) Find the market maker's profit after one week.
b) Find the cost to rebalance the portfolio after one week.

Frequent rehedging will decrease the chance that the owner of the portfolio will see a large profit or loss. Assuming that the owner of the portfolio rehedges it every $h$ years, the periodic and annual variances in the portfolio's return can be calculated using the **Boyle-Emanual formulas**. These formulas are given below.

- The periodic variance in portfolio's return is given by $\text{Var}\left[R_h\right] = 0.5\left(S^2\sigma^2\Gamma h\right)^2$.
- The annual variance in the portfolio's return is given by $\text{Var}\left[R\right] = 0.5\left(S^2\sigma^2\Gamma\right)^2 h$.

## Hedging Multiple Greeks

Notice that delta hedging works by creating a linear approximation of the change in the option price as a function of the price of the underlying stock. This provides a good approximation for small changes in the stock price over short periods of time, but it does have its limitations. In order to develop a hedging strategy that better matches the changes in the value of our option, it would be a good idea to try try to incorporate information provided by Greeks other than just delta. For instance, if we wanted to create a quadratic approximation of the option price as a function of the stock price, we could consider using gamma in our hedging strategy. If we wanted to try to capture the effect that the passing of time had on the value of our option, we could utilize theta. It turns out that we can, in fact, hedge a portfolio on several Greeks at once. The process is detailed below.

- For each Greek being hedged on, you will need a separate asset in your hedging portfolio.
- Each Greek being hedged on should have a value of zero for the entire portfolio.
- Recall that the value of a Greek for a portfolio is the sum of that Greek over all assets in the portfolio.
- One instrument in the portfolio can always be the underlying stock.
- The underlying stock has a delta of $\Delta = 1$ . All other Greeks for the stock are zero.

**Example 4.8**

You are given the following information regarding options on a stock:

- For a one-year 50-strike call on the stock, $\Delta = 0.8609$ and $\Gamma = 0.03252$ .
- For a one-year 60-strike call on the stock, $\Delta = 0.5996$ and $\Gamma = 0.03320$ .

A market maker sells 400 units of the 50-strike call. Determine the number of 60-strike calls and shares of the stock that need to be bought or sold in order to delta-gamma hedge this position.

**Example 4.9**

You are given the following information regarding options on a stock:

- For a two-year 120-strike call on the stock, $\Delta = 0.3888$ and $\Gamma = 0.014084$ .
- For a two-year 120-strike put on the stock, $\Delta = -0.5530$ .

A market maker sells 2000 units of the call option. Determine the number of put options and shares of the stock that need to be bought or sold in order to delta-gamma hedge this position.

**Example 4.10**

You are given the following information regarding options on a stock:

- For a 1-year 90-strike call on the stock, $\Delta = 0.7711$ , $\Gamma = 0.02594$ , and $\theta = -2.4$ .
- For a 1-year 100-strike call on the stock, $\Delta = 0.5486$ , $\Gamma = 0.02656$ , and $\theta = -3.34$ .
- For a 1-year 110-strike call on the stock, $\Delta = 0.3152$ , $\Gamma = 0.02645$ , and $\theta = -2.97$ .

A market maker shorts 1000 units of the 100-strike call option. Determine the number of 90-strike call options, 110-strike call options, and shares of the stock that need to be bought or sold in order to delta-gamma-theta hedge this position.

## 4.3 DELTA-GAMMA-THETA APPROXIMATION

### Taylor Series

Recall that the Taylor series for a function $f(x)$ centered at a point $a$ is given by $\sum_{i=0}^{\infty}\frac{1}{n!}f^{(n)}(x-a)^n$. Taylor series are often used to create polynomial approximations for $f(x)$ for values of $x$ sufficiently close to $a$. For example, by taking the first three terms of the Taylor Series, we obtain the second degree Taylor polynomial of $f(x)$ centered at $a$, which is given by $T_2(x) = f(a) + f'(a)(x-a) + 0.5f''(a)(x-a)^2$. This quadratic polynomial provides a reasonable approximation for $f(x)$ near $a$. This approximation is often expressed by writing $f(x+\Delta x) \approx f(a) + f'(a)\Delta x + 0.5f''(a)(\Delta x)^2$, where $\Delta x = x - a$.

One can define a Taylor series for a function of several variables as well. Although we will not provide the general formula for such a Taylor series here, we will point out that the second degree Taylor polynomial for a function $f(x,y)$ centered at $(a,b)$ is given by:

$$f(a+\Delta x, b+\Delta y) \approx$$
$$f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y + 0.5f_{xx}(a,b)(\Delta x)^2 + 0.5f_{yy}(a,b)(\Delta y)^2 + f_{xy}(a,b)\Delta x\Delta y$$

### Delta-Gamma-Theta Approximation

Assume the price of an option at time *t* is $V(S_t)$, where $S_t$ is the time *t* price of the underlying stock. Let $h$ represent a small increment in time, and let $\varepsilon = S_{t+h} - S_t$ represent the change in the price of the stock over that period of time. We can use Taylor series to develop the following approximations for $V(S_{t+h})$.

- $V(S_{t+h}) \approx V(S_t) + \Delta\varepsilon$ (Delta approximation)
- $V(S_{t+h}) \approx V(S_t) + \Delta\varepsilon + 0.5\Gamma\varepsilon^2$ (Delta-Gamma approximation)
- $V(S_{t+h}) \approx V(S_t) + \Delta\varepsilon + h\theta$ (Delta-Theta approximation)
- $V(S_{t+h}) \approx V(S_t) + \Delta\varepsilon + 0.5\Gamma\varepsilon^2 + h\theta$ (Delta-Gamma-Theta approximation(

**Example 4.11**

Assume the Black-Scholes framework holds. The price of a nondividend-paying stock is currently 65. You are given the following information regarding options on this stock:

- A *T*-year *K*-strike European call on the stock currently costs 13.40, has a delta of 0.5819, a gamma of 0.02084, and a theta of -4.8106.
- A *T*-year *K*-strike European put on the stock currently costs 16.27, has a delta of -0.4181, a gamma of 0.02084, and a theta of -1.4174.

One day later, the price of the stock has increased to 73. Use delta-gamma-theta approximation to estimate the new premiums for the call and the put.

**Example 4.12**

The current price of a non-dividend paying stock is 170. The volatility of the stock is 30% and the continuously compounded risk-free rate is 5%. A one-year 160-strike call on the stock currently costs 29.39.

One day later, the price of the stock increases to 172 and the value of the option increases to 30.76. Use delta-theta approximation to estimate theta for the call.

**Example 4.13**

The current price of a stock is 145. A put on this stock costs 10.34. The put has a delta of -0.4284, a gamma of 0.01353, and a theta of -2.0686.

After one week, the price of the stock has decreased. Using delta-gamma-theta approximation, you estimate that the value of the put is now 12.64. Determine the new stock price.

**Example 4.14**

The current price of a stock is 60. A 2-year 70-strike European call on this stock costs 2.87. The delta for the call is 0.3677 and the gamma is 0.02961. An investor purchases 1000 units of this call and delta-hedges her position.

One day later, the stock price increases to 68. Use delta-gamma-theta approximation to estimate the investor's overnight profit. Assume a continuously compounded risk-free rate of 3%.

## 4.4 ELASTICITY

Let $S$ be the current price of the stock and let $V$ be the value of an option on that stock. Assume that the stock price increases by $dS$, resulting in a change of $dV$ in the value of the option. The percentage change in the value of the stock is equal to $100(dS/S)\%$ and the percentage change in the value of the option is $100(dV/V)\%$. The elasticity of the option, denoted by $\Omega$, is the ratio of the percentage change in the value of the option as a function to the percentage change in the value of the underlying stock. In other words:

- $\Omega = \dfrac{100\,dV/V}{100\,dS/S} = \dfrac{dV/V}{dS/S} = \dfrac{dV}{dS}\dfrac{S}{V} = \dfrac{S\Delta}{V}$

This gives us the following formula for elasticity:

- $\Omega = \dfrac{S\Delta}{V}$

Several important properties of elasticity are stated below.

**Properties of Elasticity**

- $\Omega = \dfrac{S\Delta}{V}$
- $\Omega_{call} \geq 1$ and $\Omega_{put} \leq 0$
- The volatility of an option and the volatility of the underlying asset are related by the expression $\sigma_{option} = \sigma_{stock}|\Omega|$.
- The risk premium for a stock and an option are related by the expression $\gamma - r = \Omega(\alpha - r)$.
- The elasticity of a portfolio is the price-weighted average of the elasticity of its instruments.

**Example 4.15**

Assume the Black-Scholes framework applies. The current price of a stock is 210. The stock pays dividends continuously at a rate of 2% and has a volatility of 36%. The risk-free rate of interest is 6%. You are also given that $\text{Med}[S_2] = 234.51$.

Consider a of a 2-year, 240-strike European call on the stock.

a) Find the elasticity of the call.
b) Find the volatility of the call.
c) Find the continuously compounded expected yield on the call.

Consider a of a 2-year, 240-strike European put on the stock.

d) Find the elasticity of the put.
e) Find the volatility of the put.
f) Find the continuously compounded expected yield on the put.

**Example 4.16**

Assume the Black-Scholes framework applies. The current price of a nondividend-paying stock is 45. The volatility of the stock is less than 40%. The continuously compounded risk-free rate of interest is 5%. A 4-year, 40-strike call on this stock has a delta of 0.798562. Find the elasticity of this call.

**Example 4.17**

Assume the Black-Scholes framework applies. The risk-free rate of interest is 6%. The current price of a stock is 100. The stock pays dividends continuously at a rate of 2% and had a continuously compounded expected yield of 10%.

A one-year call on the stock has a premium of 16.13 and a continuously compounded expected return of 20%. Find delta for a one-year put on the stock with the same strike price as the call.

## Elasticity of a Portfolio

The elasticity of a portfolio is the price-weighted average of the elasticity of its instruments. For instance, assume that a portfolio consists of $n$ units of option A and $m$ units of option B. Denote the option prices by $A$ and $B$, and the option elasticities by $\Omega_A$ and $\Omega_B$. The elasticity of the portfolio is given by $\Omega_\pi = \dfrac{n\,A\,\Omega_A + m\,B\,\Omega_B}{n\,A + m\,B}$.

**Example 4.18**

An investor creates a portfolio by purchasing 5 units of a call and 3 units of a put.

- The elasticity of the call is 2.6.
- The elasticity of the put is -3.1.
- The elasticity of the portfolio is -0.1.
- The price of the portfolio is 152.

Determine the current prices for the call and the put.

**Example 4.19**

The current price of a stock is 80. A one-year at-the-money European call on the stock has a premium of 10.80. A one-year at-the-money European put on the stock has a premium of 7.70.

- Portfolio A consists of 20 long calls and 15 short puts. It has an elasticity of 14.1037.
- Portfolio B consists of 20 long calls and 10 long puts. It has a delta of 8.2131.

Find the elasticity of the call and the elasticity of the put.

**Example 4.20**

Let $S_t$ be the time $t$ price of a nondividend-paying stock. Assume that $\ln\left[S_3/S_0\right]$ is normally distributed with a mean of 0.105 and a standard deviation of 0.5196.

- A one-year, 110-strike call on the stock has a premium of 21.79 and a delta of 0.7168.
- A one-year, 124-strike call on the stock has a premium of 14.69 and a delta of 0.5691.
- A one-year, 140-strike call on the stock has a premium of 8.99 and a delta of 0.4089.

A portfolio consists of 8 long 110-strike calls, 15 short 124-strike calls, and 7 long 140-strike calls. Find the volatility in the price of this portfolio, as well as the continuously compounded expected yield on the portfolio.

## 4.5 THE SHARPE RATIO

**Sharpe Ratio**

- The Sharpe ratio of an asset is defined as the ratio of its risk premium to its volatility.
- The Sharpe ratios for stocks and options are defined by $\phi_{stock} = \dfrac{\alpha - r}{\sigma_{stock}}$ and $\phi_{option} = \dfrac{\gamma - r}{\sigma_{option}}$.
- The Sharpe ratio of an option always has the same absolute value as the Sharpe ratio of the underlying asset. More specifically, $\phi_{call} = \phi_{stock}$ and $\phi_{put} = -\phi_{stock}$.

**Example 4.21**

Assume the Black-Scholes framework applies. A stock has a current price of 80, pays dividends continuously at a rate of 2%, and has a volatility of 30%. A two-year 75-strike put on the option has an expected return of 40%. The continuously compounded risk-free rate of interest is 5%.

a) Calculate Sharpe ratio for the put.
b) Calculate the expected return for the stock.
c) Calculate the Sharpe ratio for the stock.

**Example 4.22**

Assume the Black-Scholes framework applies. A nondividend-paying stock is currently worth 100. The continuously compounded risk-free rate of interest is 6%. You are given the following information about a one-year at-the-money European call on this stock:

- The probability that call will be exercised is 0.6093.
- The expected price of the stock given that the option is exercised is 1.1897 times the one-year forward price of the stock.
- Delta for the call is equal to 0.6241.

Find the Sharpe ratio for the call

**Example 4.23**

Prices for a nondividend-paying stock over a period of 9 months are given in the table below.

| Month | Stock Price |
|---|---|
| 1 | 50 |
| 2 | 55 |
| 3 | 58 |
| 4 | 53 |
| 5 | 56 |
| 6 | 52 |
| 7 | 48 |
| 8 | 45 |
| 9 | 53 |

Assuming a continuously compounded risk-free rate of interest equal to 3%, estimate the Sharpe ratio for this stock.

**Example 4.24**

Assume the Black-Scholes framework applies. A certain stock has a current price of 120 and a risk premium of 9%. A European put on the stock has a premium of 17.51 and a delta of -0.2937. The volatility of the option is 80.51%. Determine the Sharpe ratio for the put.

## 4.6 GREEKS FOR BINOMIAL TREES

We can approximate $\Delta$, $\Gamma$, and $\theta$ for an option on a stock whose price is modeled using a binomial tree. The value of these Greeks will depend on the node at which they are calculated. We will use notation $(S, t)$ to denote a node at time $t$ at which the price is currently $S$.

- $\Delta(S,0) = \left[\dfrac{C_u - C_d}{Su - Sd}\right] e^{-\delta h}$
- $\Gamma(S,0) \approx \dfrac{\Delta(Su,h) - \Delta(Sd,h)}{Su - Sd}$
- $\theta(S,0)$ is found by solving the equation $C(Sud, 2h) = C(S,0) + \Delta(S,0)\epsilon + 0.5\Gamma(S,0)\epsilon^2 + 2h\theta(S,0)$, where $\epsilon = Sud - S$. This equation comes from the delta-gamma-theta approximation.