CHAPTER 5 – Exotic Options

5.1 ALL-OR-NOTHING OPTIONS

There are four types of all-or-nothing options.

- **Asset-or-Nothing Call.** Allows the holder to receive the stock without paying anything if $S_T > K$.
- **Asset-or-Nothing Put.** Allows the holder to receive the stock without paying anything if $S_T < K$.
- **Cash-or-Nothing Call.** Allows the holder to receive a fixed amount of cash if $S_T > K$.
- **Cash-or-Nothing Put.** Allows the holder to receive a fixed amount of cash if $S_T < K$.

Option	Notation	Payoff at time T	Price at time 0
Asset-or-Nothing Call (AONC)	$S \mid S > K$	S_T If $S_T > K$, 0 otherwise.	$S_0 e^{-\delta T} N(d_1)$
Asset-or-Nothing Put (AONP)	$S \mid S < K$	S_T If $S_T < K$, 0 otherwise.	$S_0 e^{-\delta T} N(-d_1)$
Cash-or-Nothing Call (CONC)	$1 \mid S > K$	1 If $S_T > K$, 0 otherwise.	$e^{-rT} N(d_2)$
Cash-or-Nothing Put (CONP)	$1 \mid S < K$	1 If $S_T < K$, 0 otherwise.	$e^{-rT} N(-d_2)$

If a cash-or-nothing option pays *c* (rather than 1) at expiration, then we will denote it by $c \mid S > K$ or $c \mid S < K$.

Relationship to Standard Options

- A *K*-strike European call is equal to $(S \mid S > K) - (K \mid S > K)$. Also note that $(S \mid S > K) = S_0 \Delta_C$.
- A *K*-strike European put is equal to $(K \mid S < K) - (S \mid S < K)$. Also note that $(S \mid S < K) = -S_0 \Delta_P$.

Example 5.1

Assume the Black-Scholes framework applies. The risk-free rate of interest is 6%. The volatility of a nondividend-paying stock is 32%. A 4-year cash-or-nothing put on this stock with a trigger price of 135 has a premium of 0.32494. Find the price of a 4-year asset-or-nothing call on this stock with a trigger price of 135.

Example 5.2

Assume the Black-Scholes framework applies. You are given the following information:

- A certain stock has a volatility of 25%.
- The volatility of a one-year 50-strike call on the stock has a volatility of 120.68%.
- A market-maker writes one unit of this call and delta hedges their position. The cost of the stock in the hedging portfolio is 34.74.

Find the price of a one-year cash-or-nothing call on the stock with a trigger price of 50.

Example 5.3

Assume the Black-Scholes framework applies.. You are giving the following information:

- A stock worth 75 pays dividends continuously at a rate of 2%.
- The stock has a volatility of 40%.
- The Sharpe ratio for the stock is 0.
- A six-month asset-or-nothing call on the stock with a trigger of 80 costs 36.65.

Calculate the probability that the asset-or-nothing call will be exercised.

5.2 GAP OPTIONS

A gap option is similar to a standard option, but involves two preset prices.

- The **Strike Price** of a gap option, K_1, is the amount paid for the stock if the option is exercised.
- The **Trigger Price** of a gap option, K_2, is the price that determines if the option will be exercised.

The payoffs and prices for gaps options are described in the table below.

Option	Payoff at time T	Price at time 0*
Gap Call	$S_T - K_1$ if $S_T > K_2$, 0 otherwise	$\text{GapCall} = S_0 e^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2)$
Gap Put	$K_1 - S_T$ if $S_T < K_2$, 0 otherwise	$\text{GapPut} = K_1 e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$

* Use K_2 when calculating d_1 and d_2 for a gap option.

Comments:

- Note that if $K_1 = K_2$, then the gap option is simply a standard option.
- Exercise on a gap option is non-elective. If the trigger condition is met, the option must be exercised.
- If $K_2 < K_1$ for a gap call, then a negative payoff will occur if $K_2 < S_T < K_1$.
- If $K_1 < K_2$ for a gap put, then a negative payoff will occur if $K_1 < S_T < K_2$.

Parity Relation: Gap options satisfy the following parity relation: $\text{GapCall} - \text{GapPut} = S_0 e^{-\delta T} - K_1 e^{-rT}$

Example 5.4

Assume the Black-Scholes framework applies. The current price of a stock is 180. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate is 6%. You are given the following information concerning options on the stock.

- A two-year, 210-strike gap call with a trigger of 200 currently costs 30.6182.
- A two-year, 190-strike gap call with a trigger of 200 currently costs 37.3276.

Find the volatility of the stock.

Example 5.5

The current price of a stock is 97. The stock pays dividends continuously at a rate of 3%. The risk-free rate of interest is 8%. A portfolio is created by purchasing 30 gap calls on the stock and selling 30 gap puts on the stock. Each of the options involved is a three-year options with a strike price of 93 and trigger price of 90. Find the value of the portfolio.

Example 5.6

Assume the Black-Scholes framework applies. You are given the following information concerning options on a stock:

- A one-year, 50-strike European put on the stock costs 10.50.
- A one-year cash-or-nothing put on the stock with a trigger of 50 costs 0.75.

Find the price of a one-year 60-strike gap put on the stock with a trigger of 50.

5.3 ASIAN OPTIONS

Arithmetic and Geometric Averages

Let $S_1, S_2, S_3, \ldots, S_n$ be prices of a stock at periodic intervals. We can average these stock prices in two ways.

- **Arithmetic Average:** $A(S) = \frac{1}{n}\sum S_i$
- **Geometric Average:** $G(S) = \sqrt[n]{\prod S_i}$

It should be noted that the inequality $G(S) \leq A(S)$ always holds.

In this section, we will use $\bar{S}$ to denote an arbitrary average that could be considered as being either arithmetic or geometric.

Asian Options

An Asian option is one in which the payoff is calculated by replacing either the strike price or the stock price with $\bar{S}$, the average price of the stock over the duration of the option. The averages are calculated using stock prices observed at the end of consecutive periods of equal length.

There are three descriptors used to classify Asian options:

- The average $\bar{S}$ can be an "Average Price" option or an "Average Strike Option".
- The average $\bar{S}$ can be either a arithmetic average or a geometric average.
- The option can be either a call or a put.

As a result, there are eight different types of Asian options that can be considered. A detailed description of how Asian options work is provided in the table below.

	Average Price Asian Option	Average Strike Asian Option
Description	• $\bar{S}$ replaces S_t in the payoff formula. • The actual value of the stock at expiration is not important, only the average. • A specific strike price is still used.	• $\bar{S}$ replaces K in the payoff formula. • No strike price is specified for the option. • The option compares S_t against the average price $\bar{S}$.
Calls	The payoff of a purchased call is • $PO = \max\left[0, \bar{S} - K\right]$	The payoff of a purchased call is • $PO = \max\left[0, S_t - \bar{S}\right]$
Puts	The payoff of a purchased put is • $PO = \max\left[0, K - \bar{S}\right]$	The payoff of a purchased put is • $PO = \max\left[0, \bar{S} - S_t\right]$
Effect of n	Value of option decreases as n increases.	Value of option increases as n increases.

Example 5.7

Prices for a stock on five different dates are provided below.

Date	Jan 1	Feb 1	Mar 1	Apr 1	May 1
Price	110	100	120	80	90

Find the payoff of the following options, all of which were purchased on January 1.

a) A four-month arithmetic average price Asian call with a strike price of 100.
b) A four-month geometric average price Asian call with a strike price of 100.
c) A four-month arithmetic average strike Asian put.
d) A four-month geometric average strike Asian put.

Example 5.8

Prices for a stock at the beginning of seven consecutive months are provided below.

Date	Jan 1	Feb 1	Mar 1	Apr 1	May 1	Jun 1	Jul 1
Price	50	56	51	x	54	y	58

The payoff for a 6-month arithmetic average strike Asian call expiring on Jul 1 is 3.5.
The payoff for a 6-month, 50-strike geometric average price Asian call expiring on Jul 1 is 4.35.

Find x and y.

Pricing Asian Options with Binomial Trees

Asian options can be priced using binomial trees by calculating the average stock price along each path from the left to the right. Since the averages will depend on the path taken, the trees must not be allowed to recombine.

Example 5.9

The price of a stock currently worth 100 is modeled using a 2-period forward tree with each period being six months. The stock pays dividends continuously at a rate of 2% and has a volatility of 30%. The continuously compounded risk-free rate is 6%. Calculate the premiums for the following options on the stock.

a) A one-year geometric average strike Asian call on the stock.
b) A one-year, at-the-money geometric average price Asian call on the stock.

Arithmetic versus Geometric Averages

Consider two Asian options which use different methods of calculating the average, but are otherwise identical. Since it is known that $G(S) \leq A(S)$, one can determine the relative value of these options by considering the role of the average in the formula for the payoff of the options.

5.4 BARRIER OPTIONS

Barrier options have a "barrier" or "trigger" value B specified along with the strike price K. Barrier options start out as being either "on" or "off", and can switch states once if the stock price hits the trigger at some point prior to their expiration. If the option is "on" at expiration, then the payoff is calculated as normal. There are two primary types of barrier options: Knock-In Options, and Knock-Out Options.

Knock-In Options

These options start out in the "off" state, but turn on if the stock price hits the barrier. We classify knock-in options based on the value of the barrier in relation to the current stock price.

- An *up-and-in* option is a knock-in option in which $B > S_0$.
- A *down-and-in* option is a knock-in option in which $B < S_0$.

Knock-Out Options

These options start out in the "on" state, but turn off if the stock price hits the barrier. We classify knock-out options based on the value of the barrier in relation to the current stock price.

- An *up-and-out* option is a knock-out option in which $B > S_0$.
- A *down-and-out* option is a knock-out option in which $B < S_0$.

Pricing Barrier Options with Binomial Trees

Barrier options can also be priced using binomial trees. As with Asian options, the option value at any given node is dependent on the path taken to the node, so the tree cannot be allowed to recombine.

Example 5.10

The price of a nondividend-paying stock currently worth 100 is modeled using a 3-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is four months. The continuously compounded risk-free rate is 6%. Calculate the prices for the following options on the stock.

a) A one-year up-and-in call with a strike of 110 and a barrier of 130.
b) A one-year up-and-out call with a strike of 70 and a barrier of 110.
c) A one-year down-an-in call with a strike of 110 and a barrier of 98.
d) A one-year up-and-in put with a strike of 120 and a barrier of 110.
e) A one-year down-and-in put with a strike of 100 and a barrier of 80.
f) A one-year down-and-out put with a strike of 120 and a barrier of 78.

Relationship to Ordinary Options

Barrier options are related to ordinary options in the following ways:

- The value of a barrier option is always less than or equal to the value of a similar ordinary option.
- A KI option and a KO option with the same parameters together equal a ordinary option.
- If $B \le K$, then an up-and-in call is equal to an ordinary call.
- If $B \ge K$, then a down-and-in put is equal to an ordinary put.

Example 5.11

The value of a nondividend-paying stock is currently 100. The one-year forward price of the stock is currently 108.33. The prices for several one-year at-the-money barrier options on the stock are given below.

Option	Barrier	Price
Up-and-out put	120	16.38
Up-and-out call	120	10.92
Down-and-out put	80	7.48
Down-and-in put	80	4.82
Down-and-in call	80	2.86
Down-and-out call	80	X

Find X.

Example 5.12

The current price of a nondividend-paying stock is 150. The continuously compounded risk-free rate of interest is 4%.

- The price for a two-year, 140-strike, up-and-in call with a barrier of 160 is 10.50.
- The price for a two-year, 140-strike, up-and-out call with a barrier of 160 is 14.30.

Find the price of a two-year down-and-in put with a strike of 130 and a barrier of 140.

Rebate Options

A rebate option is a special barrier option that pays a fixed amount if and only if the barrier is hit. The payment may occur at the time the barrier is hit or at expiration, depending on the terms of the option.

5.5 COMPOUND OPTIONS

A compound option is an option that has another option as its underlying asset. Since every compound option involves two options, we need to standardize some terminology and notation before continuing.

- The *compound option* is the "option on an option" and the *underlying option* is the underlying asset.
- The compound option has a strike price of x, and the underlying option has a strike of K.
- We will assume that the compound option is purchased at time 0.
- The compound option expires at time t_1 and the underlying option expires at time *T*, where $t_1 < T$.

There are four types of compound options:

- CallOnCall: Option to buy a call.
- CallOnPut: Option to buy a put.
- PutOnCall: Option to sell a call.
- PutOnPut: Option to sell a put.

Payoff of Compound Options

Let $V_u = V\left(S_{t_1}, K, T - t_1\right)$ be the value of the underlying option at time t_1.

- The payoff of a compound call at time t_1 is $\max\left[0, V_u - x\right]$.
- The payoff of a compound put at time t_1 is $\max\left[0, x - V_u\right]$.

Pricing Compound Options

Compound options can be priced using binomial trees. First, use the stock underlying the underlying option to calculate the risk-neutral probabilities. Then value the underlying option at each node at time t_1 in order to find the payoff of the compound option at each node. Calculate the expected payoff, and then discount to time 0 using the risk free rate.

Example 5.13

The price of a stock currently worth 100 is modeled using a 3-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is six months. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate is 5%.

Consider a 1.5 year call with a strike price of 80. Find the price of a six-month 20-strike compound put option on such a call.

Parity Relations for Compound Options

We have the following parity relationships for compound options. Assume all variables are as defined above.

- $CallOnCall - PutOnCall = Call - x e^{-r t_1}$
- $CallOnPut - PutOnPut = Put - x e^{-r t_1}$

Example 5.14

Assume the Black-Scholes framework applies. A stock currently with 80 pays dividends continuously at a rate of 2% and has a volatility of 30%. The continuously compounded risk-free rate of interest is 6%.

Consider a compound call option on a put. The underlying put expires 4 years from now and has a strike price of 90. The compound option expires in one year, has a strike price of 13, and costs 5.20.

Find the price of a compound put with the same underlying option as the compound call.

5.6 EXCHANGE OPTIONS

Exchange options provide the owner the right to exchange on asset for another upon expiration of the option. Thus, there are two assets involved in any exchange option.

- The underlying asset, which we will call *S*.
- The strike asset, which we will call *K*.

Notation for Exchange Option

We will denote the value of the assets at time 0 by S_0 and K_0. The time *T* values will be denoted S_T and K_T. We will denote the dividend rates of the two assets by δ_S and δ_K. Similarly, the volatility of the assets will be given by σ_S and σ_K. The payoff and prices for exchange options are given in the table below.

Relative Volatility

The volatility for a stock is calculated with respect to the currency in which the price of the stock is stated, and the stocks involved in an exchange option will each have their own volatility. To apply the Black-Scholes formula, we need to have a concept of how volatile the prices of the stocks are with respect to each other. Let R_S denote the annual return for the underlying asset and let R_K denote the annual return for the strike asset. We define the relative return by $R = R_S - R_K$. Let $\sigma = \sqrt{\text{Var}[R]}$. We can apply algebraic properties of the variance and covariance to obtain the expression at $\sigma^2 = \text{Var}[R] = \text{Var}[R_S + R_K] = \text{Var}[R_S] + \text{Var}[R_K] - 2\text{Cov}[R_S, R_K]$. Since $\sigma_S^2 = \text{Var}[R_S]$, and $\sigma_K^2 = \text{Var}[R_K]$, we can write this expression as $\sigma^2 = \sigma_S + \sigma_K - \text{Cov}[R_S, R_K]$. Now, let ρ denote the correlation coefficient of R_S and R_K. Then $\text{Cov}[R_S, R_K] = \rho\sigma_S\sigma_K$ and $\sigma^2 = \sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K$.

Pricing Exchange Options

Consider stocks *S* and *K*. Let S_T and K_T denote the time *T* prices of the two stocks. Assume that stock *S* has a volatility of σ_S and pays dividends at at rate of δ_S. Assume that stock *K* has a volatility of σ_K and pays dividends at a rate of δ_K. Let the correlation coefficient for the two stocks be given by ρ. Consider an exchange option with *n* shares of stock *S* as the underlying asset and *m* shares of stock *K* as the strike asset. We can use the Black-Scholes formula to price such an option as follows:

- Let $\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K}$.
- Let $d_1 = \dfrac{\ln(nS_0 / mK_0) + (\delta_K - \delta_S + 0.5\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.
- The price of the exchange call is given by $C = nS_0e^{-\delta_S T}N(d_1) - mK_0e^{-\delta_K T}N(d_2)$.
- The price of the exchange put is given by $P = mK_0e^{-\delta_K T}N(-d_2) - nS_0e^{-\delta_S T}N(-d_1)$.

Example 5.15

Assume the Black-Scholes framework applies. You are given the following information about stocks *A* and *B*.

- Stock *A* has a price of 110 and a volatility of 30%. It pays dividends at a rate of 4%.
- Stock *B* has a price of 100 and a volatility of 20%. It does not pay dividends.
- The correlation between the two stocks is 0.65.

Find the price of an option that gives the owner the right to exchange one share of Stock *B* for one share of Stock *A* two years from now.

Example 5.16

Assume the Black-Scholes framework applies. You are given the following information about stocks A and B.

- Stock A has a price of 120 and a volatility of 20%. It pays dividends at a rate of 2%.
- Stock B has a price of X and a volatility of 30%. It pays dividends at a rate of 2%.
- The correlation between the two stocks is -0.42.

Find the price of an option that gives the owner the right to exchange $120/X$ shares of Stock B for one share of Stock A one year from now.

Example 5.17

Assume the Black-Scholes framework applies. You are given the following information:

- The Euro-dollar exchange rate is \$1.50 per Euro. The volatility of this rate is 12%.
- The pound-dollar exchange rate is \$1.80 per pound. The volatility of this rate is 18%.
- The risk-free rate of interest for both Euros and pounds is 2%.
- The correlation coefficient between the returns for Euros and pounds is 0.6.

Find the price, in pounds, of an option that allows the owner to purchase 100 Euro for 74 pounds two years from now.

Parity Relation

Exchange options satisfy the following parity relation: $\text{Call} - \text{Put} = S_0 e^{-\delta_S T} - K_0 e^{-\delta_K T}$

Option Duality

Let A and B each represent assets. Consider an exchange option that allows the holder of the option to give up Asset A in exchange for Asset B at expiration. This option could be viewed in two different ways:

- It can be seen as a call in which Asset A represents the strike price and Asset B represents the underlying asset. Denote the premium of this call by $C(S=A,\ K=B)$.
- It can be seen as a put in which Asset A represents the underlying asset and Asset B represents the strike price. Denote the premium of this put by $P(S=A,\ K=B)$.

The values $C(S=A,\ K=B)$ and $P(S=A,\ K=B)$ each denote the premium of the same option. As a result, we see that $\text{Call}(S=A,\ K=B) = \text{Put}(S=B,\ K=A)$.

5.7 MAXIMA AND MINIMA

A standard call has a payoff of the form $\max[0, S_T - K]$ and a standard put has a payoff of $\max[0, K - S_T]$. If an exotic option has a payoff of the form $\max[g(S_T), f(S_T)]$ or $\min[g(S_T), f(S_T)]$, where g and f are linear functions, then the price of the option can be expressed in terms of a put or a call by using the following rules:

- $\max[A, B] = B + \max[A - B, 0] = A + \max[0, B - A]$
- $\min[A, B] = B + \min[A - B, 0] = A + \min[0, B - A]$
- $\max[kA, kB] = k\max[A, B]$ if $k > 0$
- $\min[kA, kB] = k\min[A, B]$ if $k > 0$
- $\max[-A, -B] = -\min[A, B]$
- $\max[A, B] + \min[A, B] = A + B$, and so $\min[A, B] = A + B - \max[A, B]$

Example 5.18

Assume the Black-Scholes framework applies. The current price of a nondividend-paying stock is 25. The stock has a volatility of 40%. The continuously compounded risk-free rate of interest is 4%. Consider an option that pays the owner the smaller of the following two values at the end of two years: 200, or 10 times the price of the stock at that time. Calculate the price of this option.

Example 5.19

Assume the Black-Scholes framework applies. You are given the following information regarding stocks X and Y.

- Both stocks are currently worth 75.
- Both stocks pay dividends continuously at a rate of 3%.
- Stock X has a volatility of 35%. Stock Y has a volatility of 25%.
- The correlation between the returns for the two stocks is 20%.

A financial derivative allows the purchaser to pay a premium today in exchange for their choice of one of the two stocks, to be delivered after two years. The selection of the stock will be made at the time of delivery. Calculate the premium for this option.

5.8 CHOOSER OPTIONS

A chooser option (also known as an as-you-like-it option) allows the owner to decide at time t whether he would like for the option to be a K-strike call, or a K-strike put, either of which will expire at time T. The underlying stock, strike price, exercise date, and premium are all determined when the option is purchased. The time t at which the holder of the option has to decide between a call or a put is also decided when the contract is initiated.

Pricing Chooser Options

We will now develop a formula for pricing chooser options. Notice that the value of the option at time t will be

$$V_t = \max\left[\text{Call}\left(S_t,\ K,\ T-t\right),\ \text{Put}\left(S_t,\ K,\ T-t\right)\right]$$

We can use rules from the previous section to rewrite this expression as

$$V_t = \text{Call}\left(S_t,\ K,\ T-t\right) + \max\left[0,\ \text{Put}\left(S_t,\ K,\ T-t\right) - \text{Call}\left(S_t,\ K,\ T-t\right)\right]$$

We now apply put-call parity to obtain

$$V_t = \text{Call}\left(S_t,\ K,\ T-t\right) + \max\left[0,\ K e^{-r(T-t)} - S_t e^{-\delta(T-t)}\right]$$

Again using rules from Section 5.7, we obtain

$$V_t = \text{Call}\left(S_t,\ K,\ T-t\right) + e^{-\delta(T-t)}\max\left[0,\ K e^{-(r-\delta)(T-t)} - S_t\right]$$

Notice that $\max\left[0,\ K e^{-(r-\delta)(T-t)} - S_t\right]$ is the payoff of a t-year put with a strike price of $K e^{-(r-\delta)(T-t)}$. Also note that the $\text{Call}\left(S_t,\ K,\ T-t\right)$ refers to the premium of a European call purchased at time t and expiring at time T. It follows from these observations that purchasing a chooser option is equivalent to purchasing a T-year, K-strike call, along with $e^{-\delta(T-t)}$ units of a t-year put with a strike price of $K e^{-(r-\delta)(T-t)}$. This is summarized below.

Pricing Chooser Options

Consider a T-year, K-strike chooser option. Assume the owner choices if the option will be a call or a put at time t. Then the price of the option at time 0 is given by:

- $V = \text{Call}\left(S_0,\ K,\ T\right) + e^{-\delta(T-t)}\text{Put}\left(S_0,\ K e^{-(r-\delta)(T-t)},\ t\right)$

Example 5.20

Assume the Black-Scholes framework applies. The current price of a stock is 100. The stock pays dividends continuously at a rate of 2% and has a volatility of 30%. The continuously compounded risk-free rate of interest is 6%. Find the price of a chooser option that expires in one year, and with a decision date 6 months from now.

Example 5.21

Assume the Black-Scholes framework applies. The current price of a nondividend-paying stock is 110. You are given the following information about derivatives on this stock:

- The two-year forward price on the stock is 121.88. The price of a two-year
- A two-year, 100-strike European put on the stock currently costs 9.70.
- A two-year, 100-strike chooser option on the stock with a decision date occurring in one year currently costs 34.63.

Find the price of a one-year, 95-strike European put on the stock.

5.9 FORWARD START OPTIONS

In a forward start option, the strike price is not when the option is entered into, but rather at some predetermined point in time between when the option is purchased and when it expires. At that time, the strike price is set to be equal to the current stock price times some multiplier that is determined at the time of purchase.

Pricing Forward Start Options

Consider a forward start option purchased at time 0 and expiring at time T. Assume that at some time $t \in (0, T)$ the strike price will be set at $K = c S_t$ for some known multiplier c.

We can apply the Black-Scholes formula, to determine the time t value of a forward strike call or put as follows.

- Let $d_1 = \dfrac{-\ln c + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$ and $d_2 = d_1 - \sigma\sqrt{T - t}$.
- The time t value of the forward start call is $C_{FS,t} = S_t e^{-\delta(T-t)} N(d_1) - c S_t e^{-r(T-t)} N(d_2)$.
- The time t value of the forward start put is $P_{FS,t} = c S_t e^{-r(T-t)} N(-d_2) - S_t e^{-\delta(T-t)} N(-d_1)$.

We can pull the time t values of these options back to time 0 to calculate the premiums. We do this by replacing S_t with the prepaid forward price $S_{0,t}^P = S_0 e^{-\delta t}$ and discounting the values to time 0 by multiplying by e^{-rt}. This yields the following expressions for the premiums for forward strike options.

Pricing Forward Start Options

Consider a forward start option purchased at time 0 and expiring at time T. Assume that at some time $t \in (0, T)$ the strike price will be set at $K = c S_t$ for some known multiplier c. The prices of such an options can be calculated as follows.

- Let $d_1 = \dfrac{-\ln c + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$ and $d_2 = d_1 - \sigma\sqrt{T - t}$.
- The price of a forward start call at time 0 is $C_{FS} = S_0 e^{-\delta T} N(d_1) - c S_0 e^{-r(T-t) - \delta t} N(d_2)$.
- The price of a forward start put at time 0 is $P_{FS} = c S_0 e^{-r(T-t) - \delta t} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$.

Example 5.22

Assume the Black-Scholes framework applies. A stock pays dividends continuously at a rate of 2% and has a volatility of 30%. The current price of the stock is 160. The continuously compounded risk-free rate of interest is 4%. You purchase a forward start call on this stock. The call expires in three years. After one year, the strike price of the call will be set to be equal to 110% of the price of the stock at that time. Price this option.

Example 5.23

Assume the Black-Scholes framework applies. The price of one share of a stock is currently 80. The Sharpe ratio for the stock is 0. The continuously compounded risk-free rate of interest is 3%. consider a forward start option on the stock that provides the owner with a one-year at-the-money put six months from now. The price of this option is 8.5153. Determine the volatility of the stock.