CHAPTER 6 – Monte Carlo Valuation

6.1 SIMULATING STOCK PRICES

Monte Carlo Simulation

When there is not a convenient formula available for pricing a particular derivative instrument, we can resort to using computer simulation to value the derivative. The process works like this:

1. Select a risk-neutral statistical model for the stock price. We will use the risk-neutral lognormal model.
2. Generate several simulated observations of the future stock price according to the model.
3. For each observation, calculate the payoff of the derivative.
4. Average the payoffs across all simulated observations.
5. Discount the average payoff using the risk free rate to obtain an estimate for the option price.

Generating Standard Normal Random Numbers

To generate simulated stock prices according to the lognormal model, we must first be able to generate random values according to a normal distribution. We will discuss two methods of doing so, both of which rely on first generating observations of a uniformly distributed random variable $U \sim Uniform(0,1)$. Most programming languages and software packages include methods for generating random values according to a uniform distribution.

The two methods we use for generating random observations of $Z \sim Normal(0,\ 1)$ are as follows:

1. Let $u_1,\ u_2,\ \ldots,u_{12}$ be 12 randomly generated observations of $U \sim Uniform(0,1)$. Now, let $z = \sum_{i=1}^{12} u_j - 6$. It can be shown that values of z generated in this way will have a mean of 0 and a variance of 1. By the Central Limit Theorem, they will also be approximately normal. Thus, any value z generated in this way can be considered to be an observation of $Z \sim Normal(0,\ 1)$.
2. Let u be a single observation of $U \sim Uniform(0,1)$. Let $z = N^{-1}(u)$. Then z will be an observation of $Z \sim Normal(0,\ 1)$.

Generating Lognormal Random Numbers

We will now discuss how to generate random values following a lognormal distribution with parameters *m* and *v*. Let $N \sim Normal(m,\ v^2)$ and $X = e^N \sim LogN(m,\ v^2)$. Notice that $N = m + v\,Z$, where $Z \sim Normal(0,\ 1)$.

- Let z be an observation of Z.
- Then $n = m + v\,z$ is an observation of the normally distributed random variable N.
- Thus $e^n = e^{m + v\,z}$ is an observation of the lognormally distributed variable X.

Risk-Neutral Model vs. True Model

The model we use when simulating stock prices depends on our intended goal.

- If we are interested in the true expected value of the stock, or the true expected payoff of an option, then we will use the true distribution, setting $m = (\alpha - \delta - 0.5\sigma^2)T$ and $v = \sigma\sqrt{T}$ in our lognormal model.
- If we are interested in pricing an option on the stock, then discounting is required. In this case, we make an assumption of risk-neutrality, setting $m = (r - \delta - 0.5\,\sigma^2)T$ and $v = \sigma\sqrt{T}$.

Simulating Stock Prices

Assume a stock has a current price of S_0, and that we wish to generate *N* simulated prices for the stock at time *T*. Then we will generate *N* observations of $Z \sim Normal(0, 1)$, denoted by $z_1, z_2, \ldots, z_n$. For each observation of Z, we will set $S_{T,i} = S_0 e^{m + z_i v}$. The value of *m* used depends whether we are using the true or risk-neutral model, as explained above.

Example 6.1

Assume the Black-Scholes framework applies. The current price of a stock is 50. The stock pays dividends continuously at a rate of 2% and has a continuously compounded expected yield of 10%. The volatility of the stock is 30%.

Four observations of a random variable that is uniformly distributed on the interval $(0, 1)$ are generated. The observations are 0.2420, 0.6554, 0.4207, and 0.8849.

a) Use these observations to simulate four values of the price of the stock after two years.

b) Use the four simulated stock prices to estimate the expected payoff for a two-year at-the-money European call on the stock.

Simulating a Sequence of Stock Prices

We will occasionally want to simulate a sequence of stock prices at specific intervals over a period of time, as opposed to simulating a single stock price at some specific point in time. for example, this will be necessary when using simulation to price path dependent options, such as Asian or barrier options. Such a sequence of simulated stock prices is called a **run**.

To generate a run over the interval $[0, T]$, we first split the interval into *k* subintervals, each with length *h*. Let $z_1, z_2, \ldots, z_k$ be randomly generated observations of $Z \sim Normal(0, 1)$. We create our run iteratively as follows:

$$S_h = S_0 e^{m + z_1 v}, \quad S_{2h} = S_h e^{m + z_2 v}, \quad S_{3h} = S_{2h} e^{m + z_3 v}, \ldots, \quad S_T = S_{(k-1)h} e^{m + z_k v}$$

Each run will generate one stock price at time *T* as well as a sequence of stock prices at various times along the way. If we wish to generate *N* simulated prices, it will require kN standard normal random numbers.

Example 6.2

Assume the Black-Scholes framework applies. The current price of a stock is 50. The stock pays dividends continuously at a rate of 2% and has a continuously compounded expected yield of 10%. The volatility of the stock is 30%.

Four observations of a random variable that is uniformly distributed on the interval $(0, 1)$ are generated. The observations are 0.2420, 0.6554, 0.4207, and 0.8849.

a) Use these observations to simulate a run of quarterly prices for the stock over a period of one year.

b) Calculate the payoff for a one-year arithmetic average strike Asian call for the run generated in Part (a).

c) Calculate the payoff for a one-year, at-the-money arithmetic average price Asian call for the run generated in Part (a).

6.2 MONTE CARLO VALUATION FOR OPTIONS

When we lack a convenient formula for pricing a certain type of option, we can use simulated stock prices to estimate a price for the option. The process is detailed below.

1. Generate N simulated values of the stock price at time T using a risk-neutral model. If the option in question is path dependent, then generate N runs rather than N individual prices.
2. Calculate the payoff of the option for each simulated stock price, or for each run.
3. Calculate the mean payoff of the option. Denote this by $\bar{V}$.
4. Discount the mean payoff to time 0 using the risk-free rate to obtain a simulated price of $V = \bar{V} e^{-rT}$.

This process of using simulation to estimate option premiums is called **Monte Carlo valuation**.

Example 6.3

Assume the Black-Scholes framework applies. The current price of a stock is 80. The stock pays dividends continuously at a rate of 2% and has a continuously compounded expected yield of 11%. The volatility of the stock is 35%. The continuously compounded risk-free rate of interest is 4%.

Four observations of a random variable that is uniformly distributed on the interval $(0, 1)$ are generated. The observations are 0.8152, 0.3420, 0.1626, and 0.9713.

You these observations to along with Monte Carlo simulation to estimate the price of a six-month, 90 strike European call.

Example 6.4

Assume the Black-Scholes framework applies. The current price of a nondividend-paying stock is 120. The stock has a continuously compounded expected yield of 12%. The volatility of the stock is 30%. The continuously compounded risk-free rate of interest is 5%.

An option on this stock that pays $\left[(S_1 - 120)^{1.2}, 0\right]$ at the end of one year.

Use Monte Carlo simulation to estimate the price of this option. Use the following four draws from a random variable that is uniformly distributed on the interval $(0, 1)$:

$$0.3415, \ 0.7577, \ 0.3671, \ 0.8528$$

6.3 CONTROL VARIATE METHOD

Basic Control Variate Method

If we would like to reduce the variance in the sampling distribution for our simulated values (thus reducing the margin of error in our estimate) we can use the control variate method.

- Let V be an option whose value we wish to approximate using a Monte Carlo simulation.
- Let K be an option whose value is highly correlated with that of V, and one for which we have a pricing formula. So, we know the true price of K.
- While conducting the simulation to estimate the price of V, also estimate the price of K using the same simulated stock prices. Let $\bar{V}$ and $\bar{K}$ be the price estimates generated by the simulation.
- The control variate estimate of the value of V is given by $V^* = \bar{V} + K - \bar{K}$.

Example 6.5

The current price of a nondividend-paying stock is 50. The continuously compounded risk free rate is 6%. The price of a 4-month geometric average strike Asian call on the stock is currently 4.50.

You decide to estimate the price of a 4-month arithmetic average strike Asian call on the stock using Monte Carlo simulation. You also decide to apply the basic control variate method, with the geometric average strike option being the control variable.

The results of three simulated runs for the stock are shown below.

Run	$S_{0.25}$	$S_{0.50}$	$S_{0.75}$	S_1
1	45.13	48.26	42.34	56.82
2	48.89	53.46	55.55	60.13
3	52.87	56.68	57.29	45.38

Find the simulated price for the arithmetic average strike call.

Notice that the variance of the estimate V^* is given by $\text{Var}\left[V^*\right] = \text{Var}\left[\bar{V}\right] + \text{Var}\left[\bar{K}\right] - 2\text{Cov}\left[\bar{V}, \bar{K}\right]$. If the values of $\bar{V}$ and $\bar{K}$ are highly correlated, then it will be true that $\text{Var}\left[V^*\right] < \text{Var}\left[\bar{V}\right]$, resulting in the a reduction in the variance of the estimate.

Example 6.6

Consider two similar options: Option A and Option B. Assume that there exists a pricing formula for Option B, but no such formula for Option A. Monte Carlo simulation is used to estimate the price of an Option A. The basic control variate method is used, with Option B as the control variable. You are given:

- The naive Monte Carlo estimate for the price of Option A has a variance of 7.6.
- The naive Monte Carlo estimate for the price of Option B has a variance of 5.2.
- The correlation coefficient of the two estimated prices is 0.8.

Calculate the reduction in variance obtain by using the control variate method.

Boyle Modification

Generalizing on the idea basic control variate method, assume we estimate the value of V using an expression of the form $V' = \bar{V} + \beta(K - \bar{K})$. Notice that the variance in our modified estimate V' is given by:

$$\text{Var}\left[V'\right] = \text{Var}[\bar{V}] + \beta^2 \text{Var}[\bar{K}] - 2\beta \text{Cov}[\bar{V}, \bar{K}]$$

And the overall variance reduction is equal to:

$$\text{Var}[\bar{V}] - \text{Var}\left[V'\right] = -\beta^2 \text{Var}[\bar{K}] + 2\beta \text{Cov}[\bar{V}, \bar{K}]$$

The reduction in variance is a quadratic function of β, and attains its maximum at $\beta = \dfrac{\text{Cov}[\bar{V}, \bar{K}]}{\text{Var}[\bar{K}]}$.

The **Boyle modification** V^{BM} is the estimate of the form $V^{BM} = \bar{V} + \beta(K - \bar{K})$, where β is chosen to maximize the reduction in variance.

Boyle Modification

- Let V be an option whose value we wish to approximate using a Monte Carlo simulation.
- Let K be an option whose value is highly correlated with that of V, and one for which we have a pricing formula.
- Let $\bar{V}$ and $\bar{K}$ denote the naive estimates for V and K obtained by the simulation.
- The Boyle modification estimate for V is given by $V^{BM} = \bar{V} + \beta(K - \bar{K})$, with β given by:
  $$\beta = \frac{\text{Cov}[V, K]}{\text{Var}[K]} = \frac{E[V K] - E[V]E[K]}{E\left[K^2\right] - E[K]^2} = \frac{\left(\sum v_i k_i\right) - n\bar{V}\bar{K}}{\left(\sum k_i^2\right) - n\bar{K}^2}$$
- Note that $\text{Var}\left[V^{BM}\right] = \text{Var}[\bar{V}] + \beta^2 \text{Var}[\bar{K}] - 2\beta \text{Cov}[\bar{V}, \bar{K}] = \text{Var}[\bar{V}]\left(1 - \rho^2_{\bar{V}\bar{K}}\right)$.

Example 6.7

Consider two similar options: Option X and Option Y. Assume that there exists a pricing formula for Option X, but no such formula for Option Y. Monte Carlo simulation is used to estimate the price of an Option Y. A control variate method is used, with Option X as the control variable. Let $\bar{X}$ and $\bar{Y}$ denote the naive estimates resulting from the simulation. You are given that $\text{Var}[\bar{X}] = 1.44$, $\text{Var}[\bar{Y}] = 2.56$, and $\rho_{\bar{X}\bar{Y}} = 0.63$.

Assume that the final estimate for Y will have the form $Y' = \bar{Y} + \beta(X - \bar{X})$.

a) Find the value of β which will minimize $\text{Var}\left[Y'\right]$.

b) Determine the reduction in variance obtained by using this estimate rather than the naive estimate.

Example 6.8

Let A denote the price of an arithmetic average strike Asian put on a stock. Let G be the price of an otherwise equivalent geometric average strike Asian put on the same stock. A pricing formula exists for finding G, which has a premium of 18.27. No pricing formula exists for A.

You plan on using Monte Carlo simulation to estimate the value of A, using the geometric average option as a control variable. Assume the Boyle method is used. The results of the simulation are as follows:

- The naive estimates for A and G are $\bar{A} = 15.60$ and $\bar{G} = 20.42$.
- The variances for the estimates were $\text{Var}\left[\bar{A}\right] = 1.82$ and $\text{Var}\left[\bar{G}\right] = 1.40$.
- The final estimate obtained for A is $A^{BM} = 14.31$.

Find $\text{Var}\left[A^{BM}\right]$.

Using a Calculator to Find β

Note: If you are given the observations $\left(k_i, v_i\right)$, then you can easily find β using the TI-30XS. Enter the k and v values in **L1** and **L2**, respectively. Then press **[stat]**, **[2-Var Stats]**, and select **L1** and **L2**. Then β will be the value that shows up for a for the regression line. That said, you still need know the various formulas for β in case you are asked to calculate β given information other than the actual observations.

Example 6.9

Consider two options on the same stock:

- Option A is a standard one-year, 100-strike European put on a stock.
- Option B is a special one-year, 100-strike European put whose payoff is equal to the payoff of Option A raised to the power 1.1.

The Black-Scholes formula is applied to determine that the price of the standard put is 4.60. Monte Carlo simulation is used to simulated the price of Option B. The estimate obtained has for B has the form $B^* = \bar{B} + \beta\left(A - \bar{A}\right)$, where β is chosen to minimize the variance of the estimate.

Six simulated stock prices are generated: 156.34, 87.46, 95.02, 125.68, 139.56, and 91.65.

Calculate B^* and $\text{Var}\left[B^*\right]$. Assume a continuously compounded risk free rate of 4%.

6.4 OTHER VARIANCE REDUCTION TECHNIQUES

We now discuss two methods of increasing our sample size without obtaining additional random observations.

- Assume $u_1,\ u_2,\ \ldots,\ u_n$ have been generated.

Antithetic Variates

- For each *i*, add $u_i^* = 1 - u_i$ to the sample.
- For each *i*, add $n_i^* = -n_i$ to the sample.

Stratified Sampling

- For each *i*, add $u_i^* = \dfrac{i-1}{n} + \dfrac{u_i}{n}$ to the sample.
- In effect, stratified sampling breaks the interval $[0,\ 1]$ into *n* subintervals, and then uses u_i to generate a value from the *i*th subinterval.

Example 6.10

Eight random draws of a variable that is uniformly distributed on the interval $(0,\ 1)$ are generated. These draws are:

0.70	0.93	0.48	0.35	0.47	0.89	0.08	0.97

Use stratified sampling with four subintervals to generate eight more draws.