CHAPTER 7 – Binomial Tree Models for Interest Rates

7.1 GENERAL TREE MODELS

In this lesson, we will use binomial trees to model changing short-term interest rates. We will also use these interest rate models to price bonds. We begin with a discussion of notation.

Bond Price Notation

We will use the following notation when referring to the prices of zero-coupon bonds.

- Let $P(t,T)$ denote the price determined at time *t*, and paid at time *t*, of a bond maturing for \$1 at time *T.*
- Let $P_0(t,T)$ denote the price determined at time 0, but paid at time *t*, of a bond maturing for \$1 at time *T.*

We can make the following observations about these bond prices.

- Notice that $P_0(t,T)$ is the forward price of $P(t,T)$. As such, we also denote $P_0(t,T)$ by $F_{0,T}(t,T)$.
- Interest theory concepts tell us that $P_0(t,T)=\dfrac{P(0,T)}{P(0,t)}$.
- If risk free rate is constant, then $P(0,T)=e^{-rT}$. Even if the risk free rate varies, we can use $P(0,T)$ as a present value factor. The present value of a payment of K occurring at time *T* would be $P(0,T)\cdot K$.
- These bond prices are related to spot rates as follows: $P(0,T)=\dfrac{1}{\left(1+s_T\right)^T}$.

Structure of a Binomial Tree Interest Rate Model

Binomial trees can be used to model changes in short term interest rates over time. The details of how tree rate models work are provided in the following comments.

- Each node in the tree will represent the interest rate during a period of length *h*. Typically, *h* will be 1 year. It is important to remember that in interest rate models nodes represent an entire period, not a particular moment in time.
- The process used for pricing bonds using tree rate models will be path-dependent. As such, our binomial trees must not be allowed to recombine.
- Some method will be provided to determine the magnitude of an up-move or down-move in the rates when moving from one period to the next. Risk-neutral probabilities of an up-move or a down-move will also be provided.
- Unless otherwise specified, rates will be continuously compounded. There are exceptions to this rule, however. For example, the Black-Derman-Toy model that we will study in the next section makes use of annual effective rates rather than continuously compounded rates.

Pricing Bonds Using the Binomial Tree Model

We use the following process for pricing a zero-coupon bond using a binomial tree model.

1. For each path through the binomial tree, we use the rates along that path to discount the face value of the bond back to time 0, thus obtaining one hypothetical price for each path.
2. We then calculated a weighted average of our hypothetical bond prices, with the weights determined by the probability of each path occurring. This weighted average will be our bond price.

Take note of the following comments regarding this process.

- Since each node represents an entire period rather than a single moment in time, an n period tree rate model will only fork n-1 times. For example, a three-year binomial tree rate model with one-year periods will only fork twice.
- If the rates in our model are continuously compounded, then we can add the rates at each node along a given path to get the total rate to discount by. This is not true when working with annual effective rates.
- It is not valid to calculate the "expected yield rate" for each period and then use those rates to discount the face value back to time 0. That will result in a different (incorrect) bond price.

Example 7.1

A three-period binomial tree interest rate model is constructed with each period being one year. The initial interest rate is 5%. The rate will either increase or decreased by 1.5% each period, with the risk-neutral probability of an increase being equal to 55%.

a) Determine the price of a three-year, 1000-par zero-coupon bond using this model.

b) Determine the price of a two-year, 1000-par zero-coupon bond using this model.

c) Determine the one-year forward price for a two-year 1000-par zero-coupon bond.

We can use binomial tree rate models to price options on bonds, as illustrated in the following example.

Example 7.2

A three-period binomial tree interest rate model is constructed with each period being one year. The initial interest rate is 6%. The rate will either increase or decreased by 1% each period, with the risk-neutral probability of an increase being equal to 60%.

a) Determine the price of a two-year, 950-strike European call on a one-year, 1000-par zero-coupon bond.

b) Determine the price of a two-year, 950-strike American call on a one-year, 1000-par zero-coupon bond.

7.2 THE BLACK-DERMAN-TOY MODEL

Characteristics of the Black-Derman-Toy Model

The Black-Derman-Toy model is a specific binomial rate tree model with the following characteristics.

- Each period has length h (usually 1 year).
- The rates provided are annual effective rather than continuously compounded.
- The risk-neutral probability of an up-move is $p^* = 0.5$.
- Let $r_{t,i}$ be the rate for period t node that is positioned i nodes above the bottom node.
- The rates along the bottom path $(r_{1,0}, r_{2,0}, r_{3,0}, \ldots)$ are generally provided.
- Each period will have its own short term volatility, which will be denoted by σ_t.
- Given the rate at a node $r_{t,i}$, the rate at the next node above is obtained by the formula $r_{t,i+1} = r_{t,i}e^{2\sigma_t\sqrt{h}}$.
- There are no formulas relating the rates from two different periods. However, if we know two rates from the same period, we can use the short term volatility to determine all of the other rates for that period.

Example 7.3

Use the incomplete Black-Derman-Toy interest rate model provided below to find the price of a 1000-par, three-year zero-coupon bond. Assume that each period represents one year.

		$r_{2,2} = 0.06$
	$r_{1,1} = 0.05$	
$r_{0,0} = 0.04$		$r_{2,1} = x$
	$r_{1,0} = 0.03$	
		$r_{2,0} = 0.02$

Example 7.4

You are given the following information regarding a three-period Black-Derman-Toy model.

- Each period is one year.
- $r_{0,0} = 0.05$, $r_{1,0} = 0.04$, and $r_{2,0} = 0.03$.
- $\sigma_1 = 0.20$ and $\sigma_2 = 0.25$.

Find the two-year forward price for a 1000-par, one-year zero-coupon bond.

Example 7.5

Assume that short-term rates are modeled using a Black-Derman-Toy tree model, with each period being one year. The current interest rate is 6%. The short term volatility in bond prices during the first year is 20%. The price of a two-year, 1000-par zero-coupon bond is currently 892.16. Find $r_{1,0}$ and $r_{1,1}$.

Long-Term Volatility

- Assume a bond maturing for 1 at time T is purchased at time 0 for a price of $P(0,T)$.
- The value of the bond after one year will be $P(1,T)$
- $P(1,T)$ has two possible values, which we will denote by $P_u(1,T)$ and $P_d(1,T)$.
- Let $y_u(1,T)$ be the annual effective yield rate obtained by purchasing the bond at time 1 for $P_u(1,T)$. Similarly, let $y_d(1,T)$ be the annual effective yield rate obtained by purchasing the bond for $P_d(1,T)$.
- Then $y_u(1,T) = \dfrac{1}{P_u(1,T)^{T-1}} - 1$ and $y_d(1,T) = \dfrac{1}{P_d(1,T)^{T-1}} - 1$.
- The volatility in the price of $P(1,T)$ is given by $\sigma_{1,T} = \dfrac{1}{(T-1)\sqrt{h}} \ln\left[\dfrac{y_u(1,T)}{y_d(1,T)}\right]$.

Example 7.6

Consider the following Black-Derman-Toy interest rate tree model.

		$r_{2,2} = 0.1024$
	$r_{1,1} = 0.08$	
$r_{0,0} = 0.06$		$r_{2,1} = x$
	$r_{1,0} = 0.05$	
		$r_{2,0} = 0.04$

Find the volatility in year-one for a three-year zero-coupon bond.

7.3 PRICING CAPS

Assume that an individual borrows money at a floating rate and makes interest rate payments at the end of each period. An interest rate cap is a contract that guarantees that the borrower will not have to pay more than a certain fixed rate, regardless of what the short-term rate actually is during that period. The borrower must pay a premium to obtain a cap. We can use binomial rate trees to price caps.

Throughout this section, we will use L to denote the loan amount. This is also called the **notional value**. We will also use r_k to denote the fixed interest rate cap.

Caplets

Given any node, we can calculate the value of the cap to the borrower during the period represented by that node. These values are called caplets. The first step in pricing a cap is to determine the value of the caplets at each of the nodes in the tree. The value of the caplet for the node $r_{t,i}$ will be denoted by $C_{t,i}$ and is calculated as follows:

- If $r_{t,i} \le r_k$, the the value of the caplet at this node is $C_{t,i} = 0$.
- If $r_{t,i} > r_k$, the the value of the caplet at this node is $C_{t,i} = L(r_{t,i} - r_k)$.
- In general, the value of the caplet is $C_{t,i} = \max\left[0,\ L\left(r_{t,i} - r_k\right)\right]$.
- Note that interest payments are made at the end of the period. Thus, the value found above is the value of the caplet at *the end* of the period represented by the node in question.

Pricing Caps

The value of a cap is the probability weighted sum of the present value of all caplets. The following comments are important to keep in mind when pricing caplets.

- It is necessary to discount the value of ALL of the caplets. It is not enough to work only with caplets at the terminal nodes.
- If your rate tree is drawn in so as to allow recombination of nodes, then when discounting a caplet to find its present value, you must discount along each path leading to that node.
- Since the interest payments occur at the end of the period, it is necessary to discount each caplet through its own node when finding its present value.

Example 7.7

Consider the following Black-Derman-Toy interest rate tree model.

$$r_{2,2} = 0.16$$
$$r_{1,1} = 0.11$$
$$r_{0,0} = 0.07 \qquad\qquad r_{2,1} = x$$
$$r_{1,0} = 0.06$$
$$r_{2,0} = 0.04$$

Find the price of a 3-year interest rate cap with a cap rate of 7.5% and a notional value of 2000.

Example 7.8

Consider the following Black-Derman-Toy interest rate tree model.

			$r_{3,3} = 0.2048$
		$r_{2,2} = x$	
	$r_{1,1} = 0.098$		$r_{3,2} = y$
$r_{0,0} = 0.08$		$r_{2,1} = 0.102$	
	$r_{1,0} = 0.072$		$r_{3,1} = z$
		$r_{2,0} = 0.06$	
			$r_{3,0} = 0.05$

Consider a four-year interest rate cap with a cap rate of 9% and a notional value of 1000.

a) Find the value of the year four caplet.

b) Find the price of the cap.