CHAPTER 1 – Forwards and Options

1.1 FORWARD CONTRACTS

Forward Contracts

A **forward contract** is an agreement between two individuals in which one party agrees to buy an asset from the other party for a predetermined price on a predetermined date. The price of the asset is decided upon when the contract is entered into, but is not paid until the transaction actually takes place. Some terminology relating to forward contracts is provided below.

- The **expiration date** is the date on which the actual sale will take place.
- The **forward price** is the amount that will be paid for the asset on the expiration date.
- The party obligated to purchase the asset benefits if the value increases, and is thus in a long position with respect to the underlying asset. As such, we say that the buyer has entered into a **long forward**.
- The party obligated to sell the asset benefits if the value decreases, and is thus in a short position with respect to the underlying asset. As such, we say that the seller has entered into a **short forward**.
- A **spot price** is the price of the asset on any specific date (most importantly at expiration).
- The **payoff** to either party involved in a forward contract is the value of the contract to that party on the expiration date. If the forward price is $F_{0,T}$ and the spot price at expiration is S_T, then the payoffs are:
  - **Long Forward Payoff:** $S_T - F_{0,T}$
  - **Short Forward Payoff:** $F_{0,T} - S_T$

Prepaid Forward Contracts

A prepaid forward contract is similar to a standard forward contract, except that the buyer pays the seller of the asset when the contract is entered into, as opposed to when the contract is fulfilled. As a result, the prepaid forward price of an asset, denoted by $F_{0,T}^{P}$, is equal to the present value of the forward price of the asset, $F_{0,T}$.

Forward Prices

The economic "Law of one Price" can be used to find formulas for arbitrage-free forward and prepaid forward prices. The following table summarized these prices under a variety of circumstances.

	$F_{0,T}$	$F_{0,T}^{P}$
Stock does not pay dividends	$S_0 e^{rT}$	S_0
Stock pays discrete dividends	$S_0 e^{rT} - \text{AV}(\text{Divs})$	$S_0 - \text{PV}(\text{Divs})$
Stock pays dividends continuously at a rate of δ	$S_0 e^{(r-\delta)T}$	$S_0 e^{-\delta T}$

Notice that in every row in the table above, we have that $F_{0,T}^{P} = PV\left(F_{0,T}\right)$.

Forwards on Currency

Forward contracts are occasionally used to lock in exchange rates of future currency exchanges. The formulas involved with working with currency forwards are no different from other forwards, but do tend to require some symbolic substitutions. The details are explained below.

- Imagine that we intend to use a domestic currency to purchase one unit of a foreign currency at time *T*.
- For convenience, denote the two currencies by $\$_d$ and $\$_f$.
- Since we will be "buying" the foreign currency with our domestic currency, it is helpful to think of the unit of foreign currency as the underlying asset.
- We will denote the current exchange rate by x_0 . This is the current cost of one unit of the foreign currency, expressed in units of the domestic currency. In other words, $\$_f 1 = \$_d x_0$ today.
- Assume that the domestic currency earns interest at a risk-free rate of r_d . Since this is the rate at which our money grows, this will play the role of the normal risk-free rate in our previous formulas.
- Assume that the foreign currency earns interest at a risk-free rate of r_f . This is the rate at which our "asset" grows, and will thus take the place of the dividend rate in the previous formulas.
- To determine the forward or prepaid forward price of a currency exchange, we can use the continuous dividend formulas along with the following substitutions: $S_0 = x_0$, $r = r_d$, and $\delta = r_f$.

Example 1.1

The currency exchange rate between euros and dollars is currently €$1 = \$1.14$. Assume that the current risk-free rate for dollars is 4% and the current risk-free rate for euros is 1%. Determine the forward price for a forward contract that guarantees the delivery of €1000 five years from now.

1.2 INTRODUCTION TO OPTIONS

Calls and Puts

An option is a derivative instrument that allows the owner of the option the right to "exercise" the option at a certain point in time in order to receive a payoff that is somehow determined by the price of the underlying asset at that moment. Their are many types of options, but the fundamental options are calls and puts.

- A **call option** is a type of derivative contract in which the owner of the option has the right, but not the obligation, to purchase the underlying asset for a preset price from the party who sold the option.
- A **put option** is a type of derivative contract in which the owner of the option has the right, but not the obligation, to sell the underlying asset for a preset price to the party who sold the option.

Characteristics of an Option

The following list defines several terms relevant to options.

- The purchaser or holder of the option has the right to decide whether or not to purchase the option at a predetermined time, called the **expiration date**, for a preset price, called the **strike price**.
- If the holder of the the option does decide to purchase the asset when the option expires, the we say that the option has been **exercised**, or that the holder has exercised his right to purchase the asset.
- The individual who sold the option is called the **writer** of the option. The writer of the option is obligated to sell the asset if the purchaser chooses to exercise.
- When buying an option, the purchaser must pay some amount of money to the writer of the option. That amount of money is called the **option premium**. Writers sell options to collect the premium.
- We will often denote the premium of a call as either "Call" or "C". Similarly, we will generally denote the premium of a put as either "Put" or "P". When working with an arbitrary option that could be either a call or a put, we will usually denote its premium by "V".

American vs. European Options

There are two different styles of options in common usage; European options and American options. **Europeans options** are only able to be exercised on the date of expiration for the option. **American options**, on the other hand, are able to be exercised on any date up until the expiration date.

Position with Respect to the Underlying Asset

The two parties involved in an option will hold opposite positions with respect to the underlying asset, as explained below.

- **Long (Purchased) Call**. The purchaser of a call is long with respect to the underlying asset.
- **Short (Written) Call.** The writer of a call is short with respect to the underlying asset.
- **Long (Purchased) Put**. The purchaser of a put is short with respect to the underlying asset.
- **Short (Written) Put.** The writer of a put is long with respect to the underlying asset.

Option Payoff and Profit

The **payoff** of an option for a certain party is the net gain or loss for that party. The **profit** at expiration for an option is the payoff up or down by the future value of the premium, depending on whether the party in question paid or received the premium. In the formulas for payoff and profit provided below, S_T denotes the spot price of the asset at expiration, while K denotes the strike price of the option.

- **Long Call Option:** $PO = \max\left[0\ ,\ S_T - K\right]$, Profit $= \max\left[0\ ,\ S_T - K\right] - FV\left(\text{Prem}\right)$
- **Short Call Option:** $PO = -\max\left[0\ ,\ S_T - K\right]$, Profit $= FV\left(\text{Prem}\right) - \max\left[0\ ,\ S_T - K\right]$
- **Long Put Option:** $PO = \max\left[0\ ,\ K - S_T\right]$, Profit $= \max\left[0\ ,\ K - S_T\right] - FV\left(\text{Prem}\right)$
- **Short Put Option:** $PO = -\max\left[0\ ,\ K - S_T\right]$, Profit $= FV\left(\text{Prem}\right) - \max\left[0\ ,\ K - S_T\right]$

<table>
<tr><th colspan="2">Long (Purchased) Call</th><th colspan="2">Short (Written) Call</th></tr>
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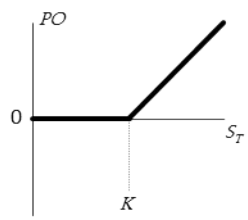
<td>
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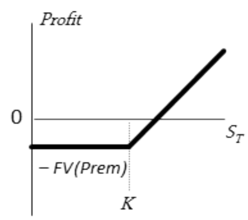
<td>
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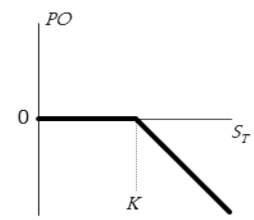
<td>
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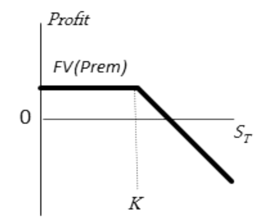
<td>
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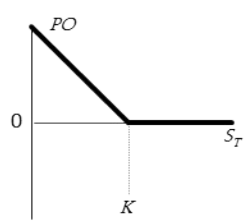
<td>
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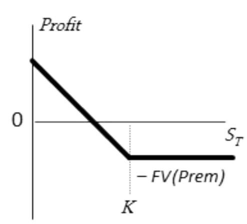
<td>
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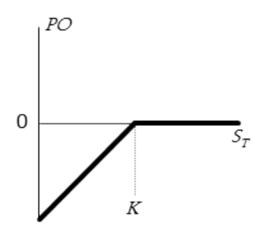
<td>
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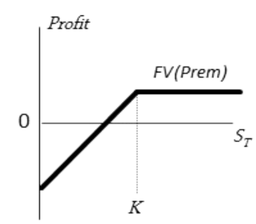
<td>
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Option Pricing

The primary goal of this course is to learn how to price options, or in other words, to calculate their premiums. Since options derive their value from some underlying asset, which is usually a stock, to be able to price an option, we much first develop a probabilistic model for the price of the underlying asset. We will work with two common stock models in this course: The binomial tree model, and the lognormal model. Each model has its advantages under certain circumstances.

As we will see later, the price of an option is usually represented as a function of the following six variables:

- Current stock price, denoted by S_0, or occasionally just S.
- The strike price, K.
- The time until expiration, T.
- The continuously compounded risk-free rate of interest, r.
- The continuously compounded dividend rate for the underlying asset, δ.
- The volatility in the price of the underlying asset, denoted by σ.

1.3 PUT – CALL PARITY

Put-Call Parity Formula

The premiums of a European call and a European with the same underlying stock and the same characteristics satisfy an important relation called put-call parity. Three versions of this relationship are stated below.

- **General Put-Call Parity.** $C(K,T) - P(K,T) = PV(F_{0,T}) - PV(K)$
  $C(K,T) - P(K,T) = F_{0,T}^{P} - PV(K)$
- **Put-Call Parity for Non-Dividend Stock:** $C(K,T) - P(K,T) = S_0 - PV(K)$

Note that the put-call parity relationship only holds for European options.

The derivation for the put-call parity relation can be found in the FM notes. We will reproduce it here, but it would be useful to review. It is important to note that the parity relation must hold regardless of the model used to price the options.

Alternate Notation. he formula for put-call parity is often written as $C(K,\ T) - P(K,\ T) = F^{P}(S) - F^{P}(K)$. Here $F^{P}(S)$ is the prepaid forward price of the stock and $F^{P}(K)$ is the prepaid forward price of a future payment of *K*, or in other words, the price of a zero-coupon bond maturing for *K*.

Example 1.2

Four portfolios are constructed using 2-year European options, all on the same stock.

- Portfolio A consists of a long 70-strike call and a short 75-strike put. Its cost is 13.03.
- Portfolio B consists of a long 70-strike put and a short 75-strike call. Its cost is -12.31.
- Portfolio C consists of a long 85-strike call and a short 90-strike put. Its cost is -1.71.
- Portfolio D consists of a long 85-strike put and a short 90-strike call. Its cost is 0.81.

Determine the continuously compounded risk-free rate of interest.

Example 1.3

The price of a stock is currently 80. The stock pays dividends continuously at a rate of 2%. You observe on the market a one-year 90-strike European call on the stock with a premium of 10.81. You also observe a one-year 90-strike European put on the same stock with a premium of 14.84. The continuously compounded risk-free rate of interest is 6%.

You determine that an opportunity for arbitrage exists. Explain how to construct the arbitrage and give the present value of the potential arbitrage profit per share of the stock.

Example 1.4

An investor is considering several *T*-year options, all on the same stock.
Let $f(K)$ denote the premium of a *K*-strike call minus the premium of a *K*+1-strike call.
Let $g(K)$ denote the premium of a *K*-strike put minus the premium of a *K*+1-strike put.
The investor notes that $f(80) = 0.54$, $g(80) = -0.38$, and $g(100) = -0.63$. Find $f(100)$.

Currency Options

In Section 1.1 we considered forward contracts on currency exchanges. It is also possible to create calls and puts on currency exchanges. The details are similar to forwards: You should think of the currency being purchased as the asset, with a current price of x_o. That currency's risk-free rate plays the role of the dividend rate, while the strike currency's risk-free rate plays the role of the risk-free rate.

When working with currency options, you will often see phrases such as "A dollar-denominated put on yen." The denomination of an option is the currency that the strike is paid in, and (usually) the currency in which the premium is paid.

Example 1.5

A one-year dollar denominated European call on pounds with a strike price of 1.60 currently sells for \$0.08. A one-year dollar denominated European put on pounds with a strike price of 1.60 currently sells for \$0.02. The continuously compounded risk-free rate for dollars is 2%, while the continuously compounded risk-free rate for pounds is 5%. Determine the current exchange rate in dollars per pound.

Option Duality

Let X and Y each represent assets. They could represent money, stocks, bonds, or any other sort of asset. They could both represent amounts of money, but perhaps in different currencies. Assume that an option allows the holder of the option to give up Asset X in exchange for Asset Y at expiration. This option could be viewed in two different ways:

- It can be seen as a call in which Asset X represents the strike price and Asset Y represents the underlying asset. We often denote calls such as this as $C(A=Y, K=X)$.
- It can be seen as a put in which Asset X represents the underlying asset and Asset Y represents the strike price. We often denote puts such as this as $P(A=X, K=Y)$.

We will study options in which the strike asset and underlying asset are both stocks (called exchange options) in some detail in Section 5.6. For now, note that since $C(A=Y, K=X)$ and $P(A=X, K=Y)$ are two ways of denoting the same option, it must be the case that $C(A=Y, K=X) = P(A=X, K=Y)$. This observation can be useful when working with options on currency exchanges.

Example 1.6

The continuously compounded risk-free rates for dollars and yen are currently 4% and 3%, respectively. A four-year yen-denominated European call on \$1 with a strike price of ¥125 currently has a premium of ¥2.2793. A four-year dollar-denominated European call on ¥1 with a strike price of \$0.008 currently has a premium of \$0.0014115. Determine the current exchange rate in yen per dollar.

Example 1.7

A financial institution knows that it will need to purchase €200,000 two years from now in order to satisfy certain obligations. The company decides to hedge against rising exchange rates and purchases options allowing them to pay no more than \$250,000 for the euros. Find the price of these options, in dollars, given the following information:

- The current exchange rate is \$1.15 per €1.
- The continuously compounded risk-free rate for dollars is 4%.
- The continuously compounded risk-free rate for euros is 3%.
- A two-year European call on \$1 with a strike of €0.80 currently costs €0.08.

1.4 BOUNDS FOR OPTIONS PREMIUMS

We will soon begin to learn methods for pricing options. In this section, however, we will discuss certain bounds that option premiums must adhere to regardless of the pricing model used to generate them.

Let C_{EUR}, P_{EUR}, C_{AM}, and P_{AM} denote the premiums for a European call, a European put, an American call, and an American put on the same underlying asset, which is currently valued at *S*. Assume all four options have a strike price of *K*, and all four expire at time *T*. The following inequalities provide bounds for these premiums.

Bounds for Option Premiums

1. C_{EUR}, P_{EUR}, C_{AM}, and P_{AM} are all greater than or equal to 0.
2. $P_{EUR} \le P_{AM}$ and $C_{EUR} \le C_{AM}$
3. $S e^{-\delta T} - K e^{-rT} \le C_{EUR} \le S e^{-\delta T}$ and $S - K \le C_{AM} \le S$
4. $K e^{-rT} - S e^{-\delta T} \le P_{EUR} \le K e^{-rT}$ and $K - S \le P_{AM} \le K$

Brief explanations for the inequalities above are given below.

1. Is it never harmful to own an option, and so an option can never have a negative premium.
2. American options carry all of the rights of European options, and thus and American option will never be cheaper than a similar European option.
3. The inequality $S e^{-\delta T} - K e^{-rT} \le C_{EUR}$ follows from put-call parity, along with the observation that $P_{EUR} \ge 0$. The inequality $C_{EUR} \le S e^{-\delta T}$ follows from the fact that you would never pay more for a European call than you would for a prepaid forward expiring at time *T*. Since American options can be exercised immediately, we can set $T = 0$ in the previous inequalities to obtain $S - K \le C_{AM} \le S$.
4. The inequality $K e^{-rT} - S e^{-\delta T} \le P_{EUR}$ follows from put-call parity, along with the observation that $C_{EUR} \ge 0$. The inequality $P_{EUR} \le K e^{-rT}$ follows from the fact that you would never pay more for a European put than you would for a bond that paid *K* at time *T*. Since American options can be exercised immediately, we can set $T = 0$ in the previous inequalities to obtain $K - S \le P_{AM} \le K$.

Arbitrage

If any of the inequalities above are violated, then opportunities for arbitrage exist. The details of constructing an arbitrage will depend on the situation, but the general idea is to buy an underpriced asset and sell an overpriced asset.

Example 1.8

The current price of a stock is 120. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate of interest is 4%. A two-year 150-strike put on the stock with a price of 21 is currently available.

An arbitrage opportunity exists by buying or selling the put, buying or selling *K* shares of the stock, and by borrowing or lending *B* dollars. Find *K* and *B* and determine what assets need to be bought and sold, and if the arbitrage involves borrowing or lending.

Time Until Expiration

Assume that $T_1 < T_2$. Since an American option expiring at time T_2 is able to be exercised at time T_1 , we have:

- $C_{AM}(S, K, T_1) \leq C_{AM}(S, K, T_2)$
- $P_{AM}(S, K, T_1) \leq P_{AM}(S, K, T_2)$

The previous statements can be summarized by stating that the value of an American option is an increasing function of *T*, the time until expiration. The longer until expiration, the greater the value of the American option.

Generally speaking, less can be said for about the relative prices of European options with different settlement dates, especially if the underlying asset pays dividends. The following two statements will hold true if the asset does NOT pay dividends.

- $C_{EUR}(S, K, T_1) \leq C_{EUR}(S, K, T_2)$. This follows from the fact that American and European calls are equivalent for non-dividend stocks. (See Section 2.2.)
- $C_{EUR}(S, K, T_1) \leq C_{EUR}(S, K e^{-r(T_2 - T_1)}, T_2)$ and $P_{EUR}(S, K, T_1) \leq P_{EUR}(S, K e^{-r(T_2 - T_1)}, T_2)$

Calls on Non-Dividend Assets

It can be shown that the value of an American call on a non-dividend paying stock at any moment is always greater than or equal to the payoff received for exercising the option at that moment. As a result, it is never optimal to exercise an American call on such a stock early. If the owner of the call wished to no longer own the call, it would be better to sell the call than exercise it.

Since early-exercise of an American call on a non-dividend stock is never optimal, such calls are actually equivalent to their European counterparts. This leads us to the following conclusion.

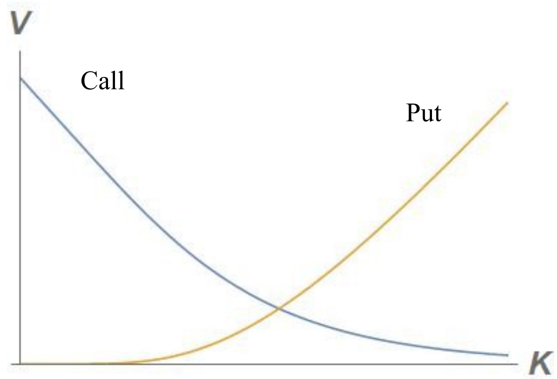
> If the underlying asset pays no dividends, then $C_{EUR} = C_{AM}$.

1.5 DIFFERENT STRIKE PRICES: MONOTONICITY

Monotonicity

The owner of a call must pay the strike price if the call is exercised. If two otherwise equivalent calls have different strike prices, the one with the higher strike will be less desirable, and will thus have a lower premium. This line of reasoning illustrates that a call premium must be a decreasing function of *K*. Since the owner of a put receives the strike price if the put is exercised, similar logic leads us to conclude that put premiums are increasing functions of *K*.

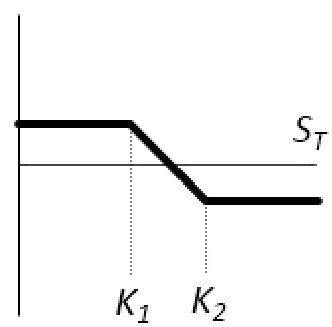
Typical plots for call and put premiums as functions of *K* are shown in figure below.



Difference in Premiums for Calls with Difference Strikes

Consider two *T*-year European calls on the same stock, one with strike K_1 and one with strike K_2, where $K_1 < K_2$. Denote the call premiums by $C_{EUR}(K_1)$ and $C_{EUR}(K_2)$. We have shown that $C_{EUR}(K_1) > C_{EUR}(K_2)$, and thus $C_{EUR}(K_1) - C_{EUR}(K_2) > 0$. We now wish to establish an upper bound for $C_{EUR}(K_1) - C_{EUR}(K_2)$.

To establish the desired result, we construct a $K_1 - K_2$ bear spread by buying the K_2-strike call and selling the K_1-strike call. The cost of the spread is $C_{EUR}(K_2) - C_{EUR}(K_1)$. Note that for a bear spread, the lowest profit at expiration occurs when $S_T > K_2$. In that case, the payoff of the spread is equal to $K_1 - K_2$, and the overall profit is equal to $K_1 - K_2 - FV\left[C_{EUR}(K_2) - C_{EUR}(K_1)\right]$. This value must be negative. If not, we would have a spread that is always profitable, which would provide an opportunity for arbitrage. Noting that the profit must be negative when $S_T > K_2$ allows us to establish the following string of inequalities:



- $0 > K_1 - K_2 - FV\left[C_{EUR}(K_2) - C_{EUR}(K_1)\right]$
- $FV\left[C_{EUR}(K_2) - C_{EUR}(K_1)\right] > K_1 - K_2$
- $C_{EUR}(K_2) - C_{EUR}(K_1) > PV\left(K_1 - K_2\right)$
- $C_{EUR}(K_1) - C_{EUR}(K_2) < PV\left(K_2 - K_1\right)$

A similar result holds for American calls with different strike prices. American options can, in theory, be exercised immediately, in which case $PV\left(K_2 - K_1\right) = K_2 - K_1$. This leaves us with $C_{AM}(K_1) - C_{AM}(K_2) < K_2 - K_1$.

The results in this subsection can be summarized by noting that the difference in the price of two call options with different strikes must be less that the difference in the strike prices. In the case of European options, we have the stronger result that the difference in call prices is less than the present value of the difference in the strike prices.

Example 1.9

Prices for three 2-year European calls with strike prices 50, 52, and 55 are given as follows: $C(50) = 5.97$, $C(52) = 3.80$, and $C(55) = 1.31$. An opportunity for arbitrage exists. Construct a spread that could take advantage of this opportunity using no more than one of each option. Find the potential range of profits for this spread. Assume a risk-free rate of 4%.

Difference in Premiums for Puts with Difference Strikes

Consider two *T*-year European put on the same stock, one with strike K_1 and one with strike K_2, where $K_1 < K_2$. Using an argument very similar to the one we employed with calls (although using a bull spread rather than a bear spread), we can show that if the puts are European, then $P_{EUR}(K_2) - P_{EUR}(K_1) < PV(K_2 - K_1)$ and if the puts are American, then $P_{AM}(K_2) - P_{AM}(K_1) < K_2 - K_1$. Compare these inequalities to the ones obtained for calls, noting the slight differences. These differences are attributable to calls being decreasing functions of *K* while puts are increasing functions of *K*.

Example 1.10

Prices for American calls and puts with strike prices 100, 104, and 110 are provided below. Determine if any opportunities for arbitrage exist. If so, state what options need to be purchased to exploit the opportunity or opportunities.

- $C(100) = 14$, $C(104) = 11$, $C(110) = 9$
- $P(100) = 7$, $P(104) = 9$, $P(110) = 16$

Bounds on Slope

Since calls are decreasing functions of *K*, the slope of the graph of *C* as a function of *K* is always negative. Stated in terms of partial derivatives, we have that $C_K = \delta C / \delta K \leq 0$. It is also possible to establish a lower bound for C_K. Whether working with either American or European calls, if $K_1 < K_2$ then $C(K_1) - C(K_2) < K_2 - K_1$. This inequality can be rewritten as $\left[C(K_2) - C(K_1)\right] / (K_2 - K_1) > -1$. This states that the average rate of change in *C* if the strike changes from K_1 to K_2 is greater than -1. Since this is true for all K_1 and K_2 where $K_1 < K_2$, the same result holds for the instantaneous rate of change, and so $C_K > -1$. Thus, we see that $-1 < C_K < 0$.

Using similar logic, we may show that for both American or European puts, the slope of the graph of *P* as a function of *K* is always less than 1. Since *P* is an increasing function of *K*, we have that $0 < P_K < 1$.

Summary of Monotonicity Results

The results that we have established relating to monotoniticy and slope in this section are summarized below.

<table>
<tr><th>Calls</th><th>Puts</th></tr>
<tr><td>For both American and European calls:
• C is a decreasing function of K.
• $-1 < C_K < 0$

If $K_1 < K_2$, then:
• $C_{AM}(K_1) - C_{AM}(K_2) < K_2 - K_1$
• $C_{EUR}(K_1) - C_{EUR}(K_2) < PV(K_2 - K_1)$</td><td>For both American and European puts:
• P is an increasing function of K.
• $0 < P_K < 1$

If $K_1 < K_2$, then:
• $P_{AM}(K_2) - P_{AM}(K_1) < K_2 - K_1$
• $P_{EUR}(K_2) - P_{EUR}(K_1) < PV(K_2 - K_1)$</td></tr>
</table>

1.6 DIFFERENT STRIKE PRICES: CONVEXITY

Convexity

Let V be the value of any standard option. It can be shown that $V_{KK} > 0$, and thus the graph of V as a function of K is concave up, or convex. The upward concavity of V allows us to say that if $K_1 < K_2 < K_3$, then $V(K_2)$ must be less than the linearly interpolated estimate obtained by using $V(K_1)$ and $V(K_3)$. More precisely:

- Since K_2 lies between K_1 and K_2, it can be written as a linear combination of the other two values, with the coefficients summing to 1.
- Let $a = \dfrac{K_3 - K_2}{K_3 - K_1}$ and $b = \dfrac{K_2 - K_1}{K_3 - K_1}$. Then $K_2 = a\,K_1 + b\,K_3$.
- The linearly interpolated (over)estimate of $V(K_2)$ would then be $a\,V(K_1) + b\,V(K_3)$.
- Convexity assures us that $V(K_2) \;<\; a\,V(K_1) + b\,V(K_3)$.

Example 1.11

The prices for American calls and puts on a stock are provided below for three difference strike prices. Determine if any opportunities for arbitrage exist with the options. For each such opportunity, construct a spread to take advantage of the opportunity.

Strike	85	100	125
Call	16	10	2
Put	4	12	20

Example 1.12

The prices for American calls a stock are provided below for three difference strike prices. Find the range of values for X that would NOT result in arbitrage opportunities.

Strike	50	X	60
Call	7	3	2

Example 1.13

Let $C(K)$ and $P(K)$ denote the premiums for a T-year, K-strike European call and put (respectively) on one share of ABC stock, which is currently worth S. Determine which, if any, of the following statements are true.

a) $0 \;\le\; C(100) - C(105) \;\le\; 5e^{-rT}$

b) $0 \;\le\; P(100) - P(105) \;\le\; 5e^{-rT}$

c) $100\,e^{-rT} \;\le\; P(100) - C(105) + S \;\le\; 105\,e^{-rT}$

d) $C(100) - 3\,C(105) + 2\,C(115) > 0$

1.7 EARLY EXERCISE OF AMERICAN OPTIONS

Early Exercise in General

When you exercise an American option early, you lose the option (which you could have sold on the market), and you receive the payoff from exercising the option. Thus, you would only ever exercise an option early if the payoff from doing so was greater than the current value of the option. More specifically:

- **Calls:** You would exercise an American call at time *t* if and only if $C_{AM}(S_t,\ K,\ T-t) \le S_t - K$.
- **Puts:** You would exercise an American put at time *t* if and only if $P_{AM}(S_t,\ K,\ T-t) \le K - S_t$.

Example 1.14

One share of XYZ stock is currently worth 67. The prices for six-month European puts on a stock with strike prices 50, 55, 60, and 65 are given in the table below.

Strike Price	50	55	60	65
Put	0.0679	0.3609	1.2276	3.0123

Six months ago, Kyle purchased four special one-year call options on XYZ stock, one for each strike price listed above. The terms of the options allow Kyle the right to exercise the calls early, but only at six months. Being at the six-month mark, Kyle has to decide which options to exercise today, and which to hold for another six months (or sell). For which of the calls would it be optimal for Kyle to exercise today?

Calls on a Non-Dividend Paying Stock

It can be shown that if the underlying stock pays no dividends, then $C_{AM}(S_t,\ K,\ T-t) \ge S_t - K$ for all values of *t*. Thus, early exercise of an American call on a **non-dividend** paying stock is never optimal. It follows that such an option is equivalent to its European counterpart, and so $C_{EUR} = C_{AM}$ if the stock does not pay dividends.

Puts on a Non-Dividend Paying Stock

In contrast with the situation for calls, there may be times when early exercise of a put on a non-dividend paying stock is optimal. As mentioned above, an American option should be exercised at time t if and only if $P_{AM}(S_t,\ K,\ T-t) \le K - S_t$. Using inequalities from Section 2.1, as well as put-call parity, we can also show that early exercise of an American put is NOT optimal if $C_{EUR}(S_t,\ K,\ T-t) \ge K - PV(K)$. It is important to note that if this last inequality fails, that tells us nothing about early exercise of the put.

Options on Dividend Paying Stocks

We can use principles already introduced to find conditions under which we can be sure that early exercise is NOT optimal. These statements are not equivalent to early exercise being optimal. They can identify certain circumstances where early exercise is NOT optimal, but if they fail, we are told nothing about early exercise.

- Early exercise of an American call on a dividend paying stock is NOT optimal if $PV(\text{Div}) \le K - PV(K)$.
- Early exercise of an American put on a dividend paying stock is NOT optimal if $K - PV(K) \le PV(\text{Div})$.

1.8 SYNTHETIC STOCKS AND TREASURIES

Synthetic Stocks

Assume that a stock pays continuous dividends. Then $C(K, T) - P(K, T) = S_0 e^{-\delta T} - K e^{-rT}$, or $S_0 = e^{\delta T} C(K, T) - e^{\delta T} P(K, T) + K e^{(\delta - r)T}$. Thus, buying one share of the stock today is equivalent to buying $e^{\delta T}$ call options, selling $e^{\delta T}$ put options, and lending $K e^{(\delta - r)T}$ (by buying a zero-coupon bond).

Synthetic Treasuries

If a stock pays continuous dividends, then we may also write the equation for put-call parity as $K e^{-rT} = S_0 e^{-\delta T} - C(K, T) + P(K, T)$. Thus, lending $K e^{-rT}$ (by purchasing a zero-coupon bond), is equivalent to buying $e^{-\delta T}$ shares of the stock, selling one call, and buying one put. In either case, the portfolio will be worth *K* at time *T*.

CHAPTER 2 – Binomial Trees

2.1 INTRODUCTION TO BINOMIAL TREES

The payoff for a put or a call is a function of the price of the underlying stock on the expiration date of the option. In order to appropriately price options, we first need construct a probabilistic model for the price of the stock. We can then use this model to determine the probability that a particular option will be exercised, as well as the expected payoff of the option. We can then use this information to price the option.

There are two commonly used stock price models: the binomial tree mode and the lognormal model. In this chapter, we will student the binomial tree model. The basic "one-period binomial tree" model that we start with is a very simplistic model, but we will see later that it serves as the building block for more complicated and more realistic stock models.

One-Period Binomial Trees

A one-step binomial tree model is described as follows.

- Let *S* be the current price of the stock. This will sometimes be denoted by S_0.
- The model covers a specific length of time. The period length is denoted by *h*, and is measured in years.
- We assume that there are only two possible values of the stock at time *h*. Either the price of the stock will increase to a value of S_u, or it will decrease to a value of S_d.
- The values S_u and S_d are sometimes stated explicitly, but are often provided as multiples of *S*. If the multipliers *u* and *d* are provided, then $S_u = S \cdot u$ and $S_d = S \cdot d$.
- The probability of an up-move is denoted by p. The probability of a down-move is equal to $q = 1 - p$.

Expected Stock Price and Expected Annual Return

Given a one-period binomial tree, we may calculate the following.

- The **expected price** of the stock after 1 period is $E\left[S_h\right] = p \cdot S \cdot u + q \cdot S \cdot d$.
- The **capital gains rate** g is the continuously compounded annual rate of growth that would cause the initial stock price of *S* to grow to the expected value $E\left[S_h\right]$. In other words, $S\, e^{gh} = E\left[S_h\right]$. It follows that $g = \frac{1}{h} \cdot \ln\left(\frac{E\left[S_h\right]}{S}\right) = \frac{\ln\left(p \cdot u + q \cdot d\right)}{h}$.
- The **continuously compounded expected annual rate of dividend growth** for the stock is denoted by δ.
- The **continuously compounded expected annual rate of return** α for the stock is equal to the capital gains rate plus the continuously compounded expected rate of dividend grown, δ. That is, $\alpha = g + \delta$.

Notice that the expected rate of return α is the expected return on investment for someone purchasing the stock. They would expect return of g due to price changes, as well as an additional return of δ due to dividend growth. This gives an total return of $\alpha = g + \delta$. For a risk-averse investor to be interested in a particular stock, they would require α to be larger than the risk-free rate r.

Example 2.1

The price of a stock is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The stock pays dividends continuously at a rate of 2% and is currently worth 50. The period for the tree is 8 months. The probability of an up-move is $p = 0.65$.

a) Determine the expected price of the stock after 8 months.

b) Find the continuously compounded expected annual return for the stock over the 8 month period.

Example 2.2

The price of a stock is modeled by a one-period binomial tree with $u = 1.15$ and $d = 0.9$. The stock pays continuous dividends at a rate of 3% and is currently worth 100. The period for the tree is 9 months. The continuously compounded expected annual return for the stock is $\alpha = 11\%$. Determine the probability of an up-move.

Capital Gains Rate Versus Mean Annual Growth Rate

Let R_h be the continuously compounded rate of growth in the stock price (ignoring dividends). The value R_h has two possible values at time *h*, and is thus a random variable. In the case of an up-move, $R_h = \ln u$. If there is a down-move, then $R_h = \ln d$. Let $\mu_h = E\left[R_h\right]$. Then $\mu_h = p \cdot \ln u + q \cdot \ln d$. We now define the **mean annual growth rate**, μ, by annualizing R_h. This gives us $\mu = \left(p \cdot \ln u + q \cdot \ln d\right) / h$. Compare this quantity with *g*:

- $g = \dfrac{\ln\left(p \cdot u + q \cdot d\right)}{h}$ and $\mu = \dfrac{p \cdot \ln u + q \cdot \ln d}{h}$

These formulas are slightly different. There is an important, but subtle, distinction between the capital gains rate and the mean annual growth rate. Read the following explanations carefully and make sure you understand the distinctions. We will generally be more interested in *g* than μ, but it is important that you understand both.

- The mean annual grown rate, μ, is the annualized expected rate of growth in the stock price.
- The capital gains rate, *g*, is the annualized growth rate that results in the expected price of the stock.

Example 2.3

The price of a non-dividend-paying stock currently worth 100 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 3 months. The probability of an up-move is $p = 0.5$.

a) Find the capital gains rate for the stock.

b) Find the mean annual growth rate for the stock price.

Expected Payoff and Return for Options

Given a binomial tree model for a stock, we can use the model to determine the expected payoff as well as the expected return for a call or put on that stock expiring at the end of the period. The details are provided below.

- Consider a *K*-strike European call and a *K*-strike European put on a stock, both expiring at the end of one period. We will denote the payoff of the call at the up-node by C_u, and the payoff of the call at the down-node by C_d. Similarly, we will denote the payoff of the put at the up and down nodes by P_u and P_d, respectively. Formulas for these quantities are given by:
    - $C_u = \max\left[0,\, S_u - K\right]$ and $C_d = \max\left[0,\, S_d - K\right]$
    - $P_u = \max\left[0,\, K - S_u\right]$ and $C_d = \max\left[0,\, K - S_d\right]$
- The expected payoffs of the options are given by $E\left[\text{Call } PO\right] = p\,C_u + q\,C_d$ and $E\left[\text{Put } PO\right] = p\,P_u + q\,P_d$.
- The continuously compounded expected annual return for a particular option is denoted by γ. It is defined by $\text{Premium} \cdot e^{\gamma h} = E\left[\text{Option } PO\right]$, or $\gamma = \dfrac{1}{h} \ln\left(\dfrac{E\left[\text{Option } PO\right]}{\text{Premium}}\right)$.

Example 2.4

The price of a non-dividend-paying stock currently worth 50 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 8 months. The probability of an up-move is $p = 0.65$. An 8-month 112-strike call has a premium of 4.42. The continuously compounded risk-free rate is 4%.

a) Determine the expected call payoff and the expected return for the call.

b) Determine the expected put payoff and the expected return for the put.

Example 2.5

The price of a non-dividend-paying stock currently worth 120 is modeled by a one-period binomial tree with $u = 1.1$ and $d = 0.9$. The period for the tree is 6 months. A 6-month 125-strike put has a premium of 6.1915 and an expected yield of 8.4911%. Find the expected return for the stock.

2.2 REPLICATING PORTFOLIOS

Given a binomial tree for a stock, it is not difficult to calculate the expected payoff for an option on that stock. The premium for the option should then be the present value of this expected payoff. The issue we face is that the rate that we should use to discount the expected payoff is γ, the expected yield of the option. If we don't already know the premium, we don't have a method of calculating γ. This requires us to develop alternate methods for pricing options. We will learn two methods: replicating portfolios and risk-neutral pricing. We will cover replicating portfolios in this section, and risk-neutral pricing in the next.

Replicating Portfolios

The method of replicating portfolios allows us to price options on a stock modeled by a binomial tree without using any probabilistic concepts. In this method, we will construct a portfolio consisting of some shares of the underlying asset, as well as some amount of borrowing of lending. The portfolio will be built so that the payoffs at time *h* are exactly the same as the option in consideration at both the upper and lower nodes. Since those are the only two "possible" payoffs for the option in the binomial tree model, we conclude that the price of the option must be the same as the price of our replicating portfolio.

The process of pricing an option using a replicating portfolio is outlined below.

- Assume a stock is modeled using a 1 period binomial tree with *S*, *r*, δ, *h*, *u*, and *d* given.
- We construct a portfolio by buying Δ shares of the stock, and investing (lending) B in risk-free bonds.
- The cost of our replicating portfolio is $\Delta S + B$.
- The portfolios value at $t = h$ is $\Delta e^{\delta h} S u + B e^{rh}$ at the up node and $\Delta e^{\delta h} S d + B e^{rh}$ at the down node.
- Let V_u and V_d be the payoffs for the option being priced at the upper and lower nodes respectively.
- For either a call or a put, we set $\Delta e^{\delta h} S u + B e^{rh} = V_u$ and $\Delta e^{\delta h} S d + B e^{rh} = V_d$. Solving this system yields $\Delta = \left(\dfrac{V_u - V_d}{Su - Sd} \right) e^{-\delta h}$ and $B = \left(\dfrac{u V_d - d V_u}{u - d} \right) e^{-rh}$.
- The price of the option is then $V = \Delta S + B$.
- Note that for a call, $\Delta \geq 0$ and $B \leq 0$. In contrast, for a put we have $\Delta \leq 0$ and $B \geq 0$.
- Let Δ_C be the number of shares in the replicating portfolio for a *K*-strike call and let Δ_P be the number of shares in the replicating portfolio for a *K*-strike put. A consequence of the previous derivations is that $\Delta_C - \Delta_P = e^{-\delta h}$.

We begin by looking at a few examples involving non-dividend-paying stocks.

Example 2.6

The price of a non-dividend-paying stock currently worth 130 is modeled by a one-period binomial tree with $u = 1.2$ and $d = 0.8$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4.5%.

a) Calculate the premium of a one-year 128-strike call on the stock.

b) Calculate the premium of a one-year 128-strike put on the stock.

Example 2.7

The price of a non-dividend-paying stock currently worth 120 is modeled by a one-period binomial tree with $u = 1.3$ and $d = 0.85$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4%. Find the strike price of a one-year call option whose replicating portfolio contains 0.5926 shares of the stock.

We will now consider an example involving dividend-paying stocks.

Example 2.8

The price of a stock currently worth 100 is modeled by a one-period binomial tree with $u = 1.3$ and $d = 0.8$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 5%.

c) Calculate the premium of a one-year 98-strike call on the stock.

d) Calculate the premium of a one-year 98-strike put on the stock.

Example 2.9

The price of a stock is modeled using a one-period binomial tree with a period of six months. The difference between the price of the stock at the upper and lower nodes is 48. The difference between the payoffs of a six-month K-strike call at the upper and lower nodes is 31. The stock pays continuous dividends at a rate of 3%. Find the number of shares in the replicating portfolio for a six-month K-strike put on the stock.

Delta-Hedging

A **market maker** is an entity that facilitates the buying and selling of stocks, as well as financial derivatives such as options. A market maker is generally not interested in making a profit by holding a position with respect to a stock. They instead make money by charging small commissions to any individual using their services to conduct a trade, whether that individual is buying an asset or selling an asset.

The ideal situation for a market maker is that whenever they sell an option, they are also able to buy an identical option. The market maker would collect commissions on both transactions and would maintain a neutral position with respect to the underlying option. When the number of options bought and sold are not equal, the market maker can make up the deficit by creating synthetic option using replicating portfolios. Replicating portfolios are constructed by buying or shorting shares of the stock and borrowing or lending some amount of money.

Example 2.10

The price of a certain stock follow a one-period binomial tree model with $u = 1.3$ and $d = 0.7$. The period for the tree is six months. The price of the stock is currently 80. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate is 5%.

A market-maker writes 100 six-month at-the-money put options on the stock. Determine the number of shares of the stock the market-maker must buy or sell to delta-hedge her portfolio.

We will discuss delta hedging in more detail in Section 4.2.

2.3 RISK NEUTRAL PRICING

The method of risk neutral pricing provides an alternative to replicating portfolios for pricing options on stocks whose prices are modeled using binomial trees. The two methods produce the same results, but each have their own advantages. For some applications, the value of delta found using the replicating portfolio method might be of interest in its own rate. An advantage of risk neutral pricing is that it tends to be easier to apply when working with a multi-period binomial tree.

Risk Neutral Pricing Method

When using risk neutral pricing, we assume that we are in a "risk-neutral" world where every investment is expected to grow at the risk-free rate. The details of the method are explained below.

- Assume that the underlying stock is modeled by a one period binomial tree with parameters S, u, d, δ, and h. Also assume that the risk-free rate r is given.
- To use the risk neutral model, we do not need to know the value of p. Recall that if we did know p, then we could be able to calculate the expected price of the stock $E\left[S_h\right] = p \cdot S \cdot u + q \cdot S \cdot d$. We could subsequently calculate the capital gains rate g using $S\,e^{gh} = E\left[S_h\right]$, and the expected yield $\alpha = g + \delta$.
- When using risk neutral pricing, we assume that the expected yield is equal to r and then calculate the **risk neutral probability** p^* consistent with this return. This amounts to solving for p^* in the equation $p^* \cdot S \cdot u + \left(1 - p^*\right) \cdot S \cdot d = S\,e^{(r-\delta)h}$.
- The quantity $E^*\left[S_h\right] = p^* \cdot S \cdot u + \left(1 - p^*\right) \cdot S \cdot d$ is called the **risk neutral expected value of the stock**.
- Solving for p^* gives the formula $p^* = \dfrac{e^{(r-\delta)h} - d}{u - d}$.
- Assume now that we wish to price an option that has values of V_u and V_d at the up and downs. The **risk neutral expected payoff** of the option is given by $E^*[PO] = p^* V_u + (1 - p^*) V_d$.
- The premium is then obtained by discounting the risk neutral expected payoff using the risk-free rate. That is: $\text{Premium} = \left[p^* V_u + (1 - p^*) V_d\right] e^{-rh}$.

Risk Neutral Pricing

- $p^* = \dfrac{e^{(r-\delta)h} - d}{u - d}$ or $p^* = \dfrac{S_0 e^{(r-\delta)h} - S_d}{S_u - S_d}$
- Risk Neutral Expected Payoff: $E^*[PO] = p^* V_u + (1 - p^*) V_d$
- $\text{Premium} = \left[p^* V_u + (1 - p^*) V_d\right] e^{-rh}$

Example 2.11 The price of a stock currently worth 100 is modeled by a one-period binomial tree with $u = 1.3$ and $d = 0.8$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 5%.

a) Use risk neutral pricing to price a one-year 98-strike call on the stock.

b) Use risk neutral pricing to price a one-year 98-strike put on the stock.

The problem we just solved is identical to the one presented in Example 2.8. Compare the results to verify that the two methods do in fact generate the same prices.

Example 2.12

The price of a stock currently worth 140 is modeled by a one-period binomial tree with $u = 1.25$ and $d = 0.85$. The probability of an up-move is $p = 0.4$. The period for the tree is 1 year. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 4.5%. Find the expected yield for a one-year 135-strike European put on the stock.

Example 2.13

The price of a nondividend-paying stock currently worth 40 is modeled by a one-period binomial tree with $u = 1.25$ and $d = 0.75$. The period for the tree is 1 year. The continuously compounded risk-free rate is 4%. A 1-year, K-strike European put on the stock currently costs 4.8237. Find K.

Example 2.14

The price of a stock is modeled using a one-period binomial tree, with the period being one-year. The ratio of the true probability of an up-move to the risk-neutral probability of an up-move is 1.15. The continuously compounded risk-free rate of return is 5%. A one-year, K-strike call has a delta of $\Delta = 0.72$. Determine the continuously compounded expected rate of return for the call.

Example 2.15

The price of a nondividend-paying stock currently worth 80 is modeled by a one-period binomial tree with $u = 1.25$ and $d = 0.75$. The period for the tree is one year. A one-year, 90-strike European call on the stock currently costs 5.93. Determine the risk-free rate of interest.

Exotic Options

Binomial trees can be used to price many exotic (i.e. nonstandard) options that we will discuss later in the course. In those cases, the strategy is similar: find the payoff of the option at each terminal node, use risk-neutral probabilities to find the (risk-neutral) expected payoff of the option at expiration, and then discount the expected payoff using the risk free rate to find the price of the option.

2.4 VOLATILITY

Introduction to Volatility

When choosing a model for the price of a stock, an analyst would want the selected model to capture traits relating to the historical or expected future behavior of the stock. One obvious feature what would want to be captured would be the capital gains rate for the stock. The capital gains rate by itself does not give us enough information to effectively select a model, however. There can be many different binomial tree models with the same g. Consider the following one-period binomial tree models, for example.

a) $h = 1$, $S_0 = 100$, $u = 1.3$, $d = 0.9$, $p = 0.5$
b) $h = 1$, $S_0 = 100$, $u = 1.5$, $d = 0.7$, $p = 0.5$

In both cases, the stock being modeled has a capital gains rate of 10%. But, these models are quite different. If we were using these models to price an option, we would get very different answers. For example, assume we were pricing at at-the-money call and an at-the-money put for these stocks. Without actually calculating the priced, see if you can determine for which model the call is more valuable, and for which model the put is more valuable.

The primary difference between these two models is that prices at the in nodes in Model 2 are further apart, making the stock following that model more prone to wild swings in the price. In other words, prices for the stock in Model 2 are more volatile than prices for the stock in Model 1. We formally define the concept of volatility as the standard deviation in the continuously compounded return for the stock.

Annual Volatility

Let R be the continuously compounded rate of growth price of the stock over the course of the next year. Stated in another way, $S_1 = S_0 e^R$. We do not currently know the value of R, so it is a random variable. We define the **annual volatility, σ,** of the stock to be the standard deviation of R. In other words, $\sigma^2 = \text{Var}[R]$. Note that since $S_1 = S_0 e^R$, we have $R = \ln(S_1 / S_0)$ and $\sigma^2 = \text{Var}[\ln(S_1 / S_0)]$.

Example 2.16

Calculate the annual volatility for each of the following one-period binomial tree models.

a) $h = 1$, $S_0 = 100$, $u = 1.3$, $d = 0.9$, $p = 0.5$
b) $h = 1$, $S_0 = 100$, $u = 1.5$, $d = 0.7$, $p = 0.5$

Periodic Volatility

The stock models we work with in this course sometimes cover a period of time less then one year. To determine the volatility of a stock following such a model, we introduce the concept of periodic volatility, and explain how it relates to annual volatility.

Consider an h-year period beginning at time 0. Let R_h be the continuously compounded periodic return for a stock during this period. In other words, $S_h = S_0 e^{R_h}$. The **periodic volatility, σ_h ,** of the stock is defined to be the standard deviation of R_h. In other words, $\sigma_h^2 = \text{Var}[R_h]$, or $\sigma_h^2 = \text{Var}[\ln(S_h / S_0)]$.

Let $n = 1/h$ be the number of h-year periods in one year. Let R_1, R_2, …, R_n be the periodic returns for the stock during each of the n periods. Let R denote the annual return. Then $S_1 = S_0 e^{R_1} e^{R_2} \cdots e^{R_n} = S_0 e^{R_1 + R_2 + \ldots + R_n} = S_0 e^R$, and thus $R = R_1 + R_2 + \ldots + R_n$. We will assume that the periodic returns, R_1, R_2 …, R_n all follow the same distribution, are independent of one-another, and all have a volatility of σ_h. Then:

$$\sigma^2 = \text{Var}[R] = \text{Var}[R_1 + R_2 + \ldots + R_n] = \text{Var}[R_1] + \text{Var}[R_2] + \ldots + \text{Var}[R_n] = n\sigma_h^2 = \frac{1}{h}\sigma_h^2$$

It follows that the relationship between periodic and annual volatility is given by $\sigma = \sigma_h / \sqrt{h}$ or $\sigma = \sigma_h \sqrt{n}$.

Example 2.17

The price of a stock is modeled using a one-period, 3-month binomial tree with $u = 0.6$ and $d = 0.4$. The probability of an up-move is 60%. Determine the annual volatility of the stock.

The most important formulas relating to annual volatility and periodic volatility are summarized below.

Annual and Periodic Volatility Formulas

- $\sigma_h^2 = \text{Var}\left[\ln\left(\frac{S_h}{S_0}\right)\right]$
- $\sigma^2 = \frac{1}{h}\text{Var}\left[\ln\left(\frac{S_h}{S_0}\right)\right]$
- $\sigma = \frac{\sigma_h}{\sqrt{h}}$

Modeling Volatility

Given a binomial tree model, we have seen how to calculate the volatility of the stock being modeled. We now turn our attention to determining how to create a model that has a certain desired volatility. In particular, given σ, we want to determine values of u, d, and p that will result in a model with a volatility of σ. In fact, we will see that there are many possible values that could achieve this goal.

Consider a 1 period, h-year binomial tree. Assume, for convenience, that $p = 0.5$. Let R_h be given by $S_h = S_h e^{R_h}$. Then $R_h = \ln(S_h / S_0)$ and either $R_h = \ln(u)$ or $R_h = \ln(d)$, each with a probability of 50%. This gives us $E[R_h] = \frac{\ln u + \ln d}{2}$ and $E[R_h^2] = \frac{(\ln u)^2 + (\ln d)^2}{2}$. We can substitute into $\sigma^2 = \frac{1}{h} \cdot \text{Var}[R_h] = \frac{1}{h} \cdot \left[E[R_h^2] + E[R_h]^2 \right]$ to obtain $\sigma^2 = \frac{1}{h}\left[\frac{(\ln u)^2 + (\ln d)^2}{2} - \left(\frac{\ln u + \ln d}{2}\right)^2\right] = \frac{1}{h}\left[\frac{(\ln u - \ln d)^2}{4}\right] = \frac{\ln(u/d)^2}{4h}$. Taking square roots of both sides yields $\sigma = \frac{\ln(u/d)}{2\sqrt{h}}$, and solving for $\frac{u}{d}$ gives us $\frac{u}{d} = e^{2\sigma\sqrt{h}}$.

This does not give give us specific values for u and d, but rather a condition that the two value have to satisfy. We will now provide several commonly used models that satisfy this condition.

Binomial Tree Models

- **Forward Tree (Standard/Usual Method)**
  $u = e^{(r-\delta)h + \sigma\sqrt{h}}$ and $d = e^{(r-\delta)h - \sigma\sqrt{h}}$
- **Lognormal Tree (Jarrow-Rudd Tree)**
  $u = e^{(r-\delta-0.5\sigma^2)h + \sigma\sqrt{h}}$ and $d = e^{(r-\delta-0.5\sigma^2)h - \sigma\sqrt{h}}$
- **Cox-Ross-Rubinstein Tree**
  $u = e^{\sigma\sqrt{h}}$ and $d = e^{-\sigma\sqrt{h}}$

The Forward Tree model is sometimes called the "Usual Method in McDonald". We will use this model much more frequently than the other two. Note that we assumed $p = 0.5$ in the discussion above. If p has a different value, then these models will not result in a volatility exactly equal to σ. However, if p is not far from 0.50, the resulting volatility should be close to σ.

Example 2.18

Construct a one-period binomial forward tree using the following parameters:

- $h = 1$
- $S_0 = 80$
- $\sigma = 30\%$
- $r = 5\%$
- $\delta = 3\%$

The continuously compounded expected return on the stock is 12%. Find the continuously compounded expected annual return on a one-year at-the-money put on the stock.

Example 2.19

The price of a stock is modeled by a one-period Cox-Ross-Rubinstein tree with a volatility of 25%. The period for the tree is three months. The stock currently has a price of 200 and pays dividends continuously at a rate of 3%. The risk-free rate is 5%.

A three-month, 210-strike call on the stock has a continuously compounded expected annual yield of 42%. Determine the continuously compounded expected return for the stock.

Arbitrage

It can be shown that if a binomial tree model does not satisfy the inequality $d < e^{(r-\delta)h} < u$, then there will be opportunities for arbitrage. Of the three models we have considered, only the forward tree model guarantees that this no-arbitrage condition is met. Under "usual" circumstances, the Cox-Ross-Rubinstein and Lognormal trees will also satisfy the requirement, but there is no guarantee that these models will satisfy the criteria in general.

Historical Volatility

A common, although not always valid, assumption in the stock models we consider in the course is that the returns for a stock during any two non-overlapping segments of time with equal length will follow the same distribution. For example, let R_1, R_2, …, R_n be the periodic returns for a stock during n consecutive h-year periods., and let r_1, r_2, …, r_n denote the observed values of these returns. We will assume that each R_i follows the same distribution. We can use the observed returns r_i to calculate the sample standard deviation $\hat{\sigma}_h$ of the returns, which we will use as an estimate of the periodic volatility σ_h. We can then use the value $\hat{\sigma} = \hat{\sigma}_h / \sqrt{h}$ as an estimate for the annual volatility in the stock. This quantity is called the **historical volatility** of the stock since it is calculate using past data. A summary of the formulas relating to historical volatility are provided below.

Historical Volatility

- Let $S_{0,}$ S_1, S_2, …, S_n be the prices of a stock observed at h-year intervals .
- Let $r_i = \ln\left(S_i / S_{i-1}\right)$ be the periodic return during the ith period.
- The historical volatility is given by $\hat{\sigma} = \dfrac{\hat{\sigma}_h}{\sqrt{h}}$, where $\hat{\sigma}_h = \sqrt{\dfrac{1}{n-1}\sum\left(r_i - \bar{r}\right)^2}$.

We can use the TI-30XS to find the estimate $\hat{\sigma}_h$. The process is as follows:

- Press the **[data]** button and then enter the values of r_i into the list **L1**.
- Press **[2nd]**, **[stat]**, and then select "**1-Var Stats**".
- Select "**L1**" for Data and "**ONE**" for **FRQ**.
- Then $\hat{\sigma}_h = Sx$.
- Note that the value given for $\bar{x}$ is an estimate for the mean return μ_h, and not the capital gains rate g_h .

Example 2.20

The prices for a stock at the end of each of the first seven months of the year are given in the table below. Use these prices to estimate the annual volatility for the stock.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul
Stock Price	80	88	92	82	85	78	84

Example 2.21

The prices for a stock at the end of each of the first six months of the year are given in the table below. The historical annual volatility based on these prices is 41.05%. Find K.

Month	Jan	Feb	Mar	Apr	May	Jun
Stock Price	K	112	106	95	112	K

2.5 MULTI-PERIOD BINOMIAL TREES

As mentioned previously, one-period binomial trees are not a particularly realistic model for stock prices. We can obtain more realistic models by expanding upon the idea and considering multi-period binomial trees. The details of the multi-period binomial tree model are explained below.

- Assume that the length of time covered by the model is T years, and that this interval of time is split into n periods of length h.
- The initial stock price is S. Prices at later nodes are denoted using subscripts indicating the number of up and down moves required to reach that node.
- The probability of an up-move for any given period is p. In the case of an up-move, the price is multiplied by a factor of u. The multiplier for a down-move is d.

Pricing European Options Using Multi-Period Binomial Trees

We will explain how to price a European Option using a two-period binomial tree. The process for using a binomial tree with more than two periods is a natural extension to this method.

1. Assume that we are pricing a $2h$-year K-strike European option.
2. Denote the payoffs of the option at each of the three terminal nodes by V_{uu}, V_{ud}, and V_{dd}.
3. Use the payoffs V_{uu} and V_{ud} to calculate the price of a h-year K-strike option of the same type, sold at time h, assuming that an up-move occurred during the first period. Denote this quantity by V_u.
4. Use the payoffs V_{ud} and V_{dd} to calculate the price of a h-year K-strike option of the same type, sold at time h, assuming that a down-move occurred during the first period. Denote this quantity by V_d.
5. Use the values V_u and V_d to determine the price of the option, V.

An alternate (and equivalent) method would be to calculate the risk-neutral expected payoff at time $2h$ using $E^*\left[S_{2h}\right] = \left(p^*\right)^2 S_{uu} + 2\,p^* q^* S_{ud} + \left(q^*\right)^2 S_{dd}$, and then discount to time 0 using the risk-free rate: $V = E^*\left[S_{2h}\right] e^{-2rh}$.

Example 2.22

The price of a stock currently worth 160 is modeled by a two-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is 6 months. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 6%.

a) Calculate the premium for a one-year 185-strike European call on the stock.

b) Calculate the premium for a one-year 185-strike European put on the stock.

Example 2.23

The price of a dividend-paying stock currently worth 100 is modeled using a 3-period binomial tree with $u = 1.1$ and $d = 0.9$. Each period is one year. The continuously compounded risk-free rate of interest is 8%. The price of a three-year 120-strike European call on the stock is 3.4117. Find the price of a three-year 100-strike European call on the stock.

Example 2.24

The price of a nondividend-paying stock currently worth 100 is modeled using a 12-period binomial tree using the usual method in McDonald. The volatility of the stock is 35%. The continuously compounded risk-free rate is 6%. Find the premium for a one-year 135-strike European call on the stock.

Pricing American Options Using Multi-Period Binomial Trees

Recall that American options and European options differ in that European options can only be exercised when the option expires, whereas an American option can be exercised at any point prior to the expiration date for the option. We can use multi-period binomial trees to price American options by making a small adjustment to the process using for European options. We explain the process for a two period binomial tree below.

1. Assume that we are pricing a $2h$-year K-strike American option. We also assume that the option in question can only be exercised at the end of an h-year period.
2. Denote the payoffs of the option at each of the three terminal nodes by V_{uu}, V_{ud}, and V_{dd}.
3. We now calculate V_u. The process is more complicated than with European options.
   a) Use the payoffs V_{uu} and V_{ud} to find the price of a h-year K-strike option of the same type, sold at time h, assuming that an up-move occurred during the first period. Denote this quantity by MV_u.
   b) Find the payoff for the option at the up-node if it were exercised early. Denote this by PO_u.
   c) Let $V_u = \max\left[MV_u, PO_u\right]$.
4. We now calculate V_u.
   a) Use the payoffs V_{ud} and V_{dd} to find the price of a h-year K-strike option of the same type, sold at time h, assuming that a down-move occurred during the first period. Denote this quantity by MV_d.
   b) Find the payoff for the option at the down-node if it were exercised early. Denote this by PO_d.
   c) Let $V_d = \max\left[MV_d, PO_d\right]$.
5. Use the values V_u and V_d to determine the price of the option, V.

Example 2.25

The price of a stock currently worth 160 is modeled by a two-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is 6 months. The stock pays dividends at a continuous rate of 2%. The continuously compounded risk-free rate is 6%.

a) Calculate the premium for a one-year 185-strike American call on the stock.

b) Calculate the premium for a one-year 185-strike American put on the stock.

c) Find the value of delta for a one-year 185-strike American call on the stock.

d) Find the value of delta for a one-year 185-strike American put on the stock.

Example 2.26

The price of a nondividend-paying stock currently worth 100 is modeled by a four-period binomial tree with $u = 1.1$ and $d = 0.9$. Each period is one year. The continuously compounded risk-free rate is 4%. A four-year 90-strike American call on the stock is priced using this binomial tree model.

The price of the stock decreases during each of the first two years. Find the value of delta for the the call at the end of the second year.

Example 2.27

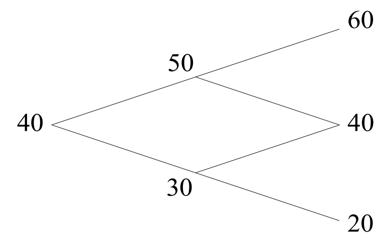
The price of a paying stock currently worth 125 and paying continuous dividends at a rate of 2% is modeled by a two-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is six months. The continuously compounded risk-free rate is 8%. You purchase a one-year K-strike put on the stock. During the first six months, the price of the stock declines. Determine the smallest value of K for which early exercise would be optimal at the end of the first six months.

Although it is not common to do so, it is possible to construct a multi-period binomial tree in which each the one-period subtrees have different values for u and d. Options on a stock modeled by such a tree can be priced using standard techniques, although a new risk-neutral probability will have to be calculated for each subtree.

Example 2.28

The price of a stock is modeled using the two-period binomial tree on the right. Each period is six months. The stock pays continuous dividends at a rate of 2%. The risk-free rate of interest is 8%.

Calculate the premium for a one-year 50-strike European put on the stock.



2.6 UTILITY

Introduction to Utility

- Let r and δ be the risk-free rate and the dividend rate, both stated as annual **effective** rates.
- Let W_u and W_d represent the "worth" of an additional dollar at the up node and at the down node, respectively.
- Define $W_H = \dfrac{p^*}{p}$ and $W_L = \dfrac{1-p^*}{1-p}$.
- Then $pW_H + (1-p)W_L = 1$.
- Then $p\left(\dfrac{1}{1+r}W_H\right) + (1-p)\left(\dfrac{1}{1+r}W_L\right) = \dfrac{1}{1+r}$.
- Let $U_H = \dfrac{1}{1+r}W_H$ and $U_L = \dfrac{1}{1+r}W_L$. Then $pU_H + (1-p)U_L = \dfrac{1}{1+r}$.
- Let $Q_H = pU_H$ and $Q_L = (1-p)U_L$. Then $Q_H + Q_L = \dfrac{1}{1+r}$.

Pricing using utility:

- $Q_H = \dfrac{p^*}{1+r}$ and $Q_L = \dfrac{1-p^*}{1+r}$.
- $S = [Q_H S_u + Q_L S_d](1+\delta)$
- $V = Q_H V_u + Q_L V_d$

CHAPTER 3 – The Black-Scholes Framework

3.1 THE LOGNORMAL DISTRIBUTION

Let R_t be the continuously compounded periodic return for a stock during from time 0 to time t. In other words, $S_t = S_0 e^{R_t}$. Notice that at time 0, neither R_t or S_t have been observed, and are thus both random variables. These variables are obviously closely linked in that if we know the value of one, we will know the value of the other. They will almost certainly follow different probability distributions, however.

Under the lognormal stock model, we will assume that the return R_t follows a normal distribution. As a consequence, S_t follows a distribution known as the lognormal distribution. Before discussing this stock model in detail, we should cover some basic facts relating to the normal and lognormal distribution.

Standard Normal CDF

Let $Z \sim \text{Normal}(0,1)$ be the standard normal distribution. The cumulative distribution function (CDF) for the normal distribution is given by $N(z) = \text{Prob}[Z \le z]$. It should be noted that $\Phi(z)$ is traditionally used to denote the standard normal CDF, and our use of $N(z)$ is somewhat nonstandard.

Let $X \sim \text{Normal}(m, v^2)$ be a normally distributed variable with mean m and standard deviation v^2. Values for the CDF of X can be calculated using the standard normal CDF as follows: $\text{Prob}[X \le x] = N\left(\dfrac{x-m}{v}\right)$.

An equation that is often useful when working with the standard normal distribution is $N(-z) = 1 - N(x)$.

We will also make frequent use of the inverse standard normal CDF, $N^{-1}(p)$. This function is defined as follows: $N^{-1}(p) = z$ if and only if $N(z) = p$.

Definition of Lognormal Distribution

Let $X \sim \text{Normal}(m, v^2)$ be a normally distributed random variable with mean m and standard deviation v. Let $Y = e^X$. Then we say that Y has a **lognormal distribution** defined by the parameters m and v. Several properties of the lognormal distribution are stated below without proof. The pdf of the distribution is also shown.

Properties of Lognormal Distribution

- $E[Y] = e^{m + 0.5v^2}$
- $\text{Var}[Y] = \left(E[Y]\right)^2\left[e^{v^2} - 1\right] = e^{2m + v^2}\left(e^{v^2} - 1\right)$
- $\text{Med}[Y] = e^m$
- $\text{Mode}[Y] = e^{m - v^2}$

Lognormal Probability Calculations

Let $X \sim \text{Normal}(m, v^2)$ and $Y = e^X$. We can calculate $\text{Prob}[Y \le k]$ as follows:

$$\text{Prob}[Y \le k] = \text{Prob}[e^X \le k] = \text{Prob}[X \le \ln k] = \text{Prob}\left[Z \le \frac{\ln k - m}{v}\right] = N\left(\frac{\ln k - m}{v}\right)$$

Example 3.1

Let X be normally distributed with a mean of 3 and a standard deviation of 2 and let $Y = e^X$.

a) Find $P[Y \le 100]$.

b) Find the 10th, 50th, and 90th percentiles of X.

c) Find the 10th, 50th, and 90th percentiles of Y.

Example 3.2

Assume that X is normally distributed with mean 0.2 and standard deviation 0.3 and let $Y = 50e^X$. Find $P[Y \le 35]$.

Constant Multiples of a Lognormal Random Variable

Let $X \sim \text{Normal}(m, v^2)$ be a normally distributed random variable with mean m and standard deviation v. Let $Y = e^X$. Then, by definition, Y is a lognormal random variable. Now, let $Z = CY = Ce^X$, where C is a constant. Then $Z = Ce^X = e^{\ln C} e^X = e^{\ln C + X}$. Let $W = \ln C + X$. Since X is normally distributed with mean m and standard deviation v, W is also normally distributed, with mean $m + \ln C$ and standard deviation v. Since $Z = e^W$, we can conclude that Z is lognormally distributed.

In summary, if Y is a lognormal random variable, then $Z = CY$ will also follow a lognormal distribution, though with different parameters from Y.

3.2 THE LOGNORMAL STOCK MODEL

In this section, we see how to construct a stock model using the lognormal distribution. Our models will be of the form $S_t = S_0 e^{R_t}$, with the *t*-year return R_t normally distributed. Let *m* and *v* be the mean and standard deviation for R_t. The question we need to answer is the following: To what should the parameters *m* and *v* be set so as to produce certain desired properties of the stock model? In particular, we would like to be able to specific the capital gains rate and the volatility of the stock being modeled.

Lognormal Parameters

Let S_0 be the current price of a stock for which we would like to construct a model. Assume that the stock pays continuous dividends at a rate of δ and is expected to have a capital gains rate of *g* and a volatility of σ. We wish to use the lognormal distribution to make predictions about value of S_t. To that end, we will set $S_t = S_0 e^{R_t}$ and assume that R_t is normally distributed. We will now derive expressions for the parameters *m* and *v* for R_t.

Let $S_t = S_0 e^{R_t}$ where $R_t \sim \text{Normal}(m, v^2)$. Then $\dfrac{S_t}{S_0} \sim LogN(m,\ v^2)$.

- **Finding *v*.** Recall from Section 2.4 that $\sigma^2 = \dfrac{1}{t}\text{Var}\left[\ln\left(\dfrac{S_t}{S_0}\right)\right]$. It follows that $\sigma^2 = \dfrac{1}{t}\text{Var}[R_t] = \dfrac{1}{t}v^2$. This tells us that $v^2 = \sigma^2 t$ and $v = \sigma\sqrt{t}$.
- **Finding *m*.** Recall from Section 2.1 that $g = \dfrac{1}{t}\cdot\ln\left(E\left[\dfrac{S_t}{S_0}\right]\right)$. Using the formula provided in Section 3.1 for the expected value of a lognormally distributed variable, we obtain $g = \dfrac{1}{t}\ln\left(e^{m + 0.5v^2}\right) = \dfrac{1}{t}\left(m + 0.5v^2\right)$. Substituting in the previously obtained expression for *v* and solving for *m*, we get $m = \left(g - 0.5\sigma^2\right)t$. Since $g = \alpha - \delta$, *m* can be written as $m = \left(\alpha - \delta - 0.5\sigma^2\right)t$.

Relationships Between Different Growth Rates

The values g, δ, α, m, and μ are closely related and easy to confuse. The following discussion aims to clarify the difference and the relationships between these quantities.

- g is the **capital gains rate**. It satisfies the expression $S_0 e^{gt} = E[S_t]$.
- δ is the continuously compounded **dividend rate** for the stock.
- α is the **expected annual yield**. It includes growth due to price changes and dividends: $\alpha = g + \delta$
- μ is the **mean annual return**, defined as $\mu = E[R]$. For the lognormal model, we have $\mu = \alpha - \delta - 0.5\sigma^2$.
- m is the **mean *t*-year** return, defined by $m = E[R_t]$. We know that $m = \left(\alpha - \delta - 0.5\sigma^2\right)t$, or $m = \mu t$.

The formulas for the parameters *m* and *v* are summarized below.

Summary of Lognormal Parameters

Let $S_t = S_0 e^{Rt}$ where $Rt \sim \text{Normal}(m, v^2)$. Then:

- $m = \left(\alpha - \delta - 0.5\sigma^2\right)t$, $m = \left(g - 0.5\sigma^2\right)t$, and $m = \mu t$ where $\mu = \alpha - \delta - 0.5\sigma^2$
- $v = \sigma\sqrt{t}$

Example 3.3

The current price for one share of a certain stock is 80. The stock pays dividends continuously at a rate of 3% and has an expected yield of 11%. The volatility of the stock is 20%. The risk-free rate is 5%. Assume the stock follows the lognormal model.

a) Find the expected price of the stock after one year..

b) Find the median price of the stock after one year.

c) Find the probability that the price of the stock after 1 year is greater than 110.

d) Find the probability that the price of the stock after 2 years is greater than 110.

e) Find the 10th percentile of the price of the stock after 1 year.

f) Find the 10th percentile of the price of the stock after 2 years.

Methods of Stating Volatility

Exam problems will state the volatility of a stock in many indirect ways. Some common methods are given below.

Methods of Stating Volatility

- $\text{Var}\left[\ln\left(S_T / S_0\right)\right] = \sigma^2 T$
- $\text{Var}\left[\ln\left(S_T\right)\right] = \sigma^2 T$
- $\text{Var}\left[\ln\left(F_{0,T}\right)\right] = \text{Var}\left[\ln\left(F_{0,T}^P\right)\right] = \sigma^2 T$
- $\ln\left[\dfrac{E\left[S_T\right]}{\text{Med}\left[S_T\right]}\right] = 0.5\sigma^2 T$

Confidence and Prediction Intervals

Assume that $R_t \sim \text{Normal}\left(m, v^2\right)$ and $S_t = S_0 e^{A t}$. Let $0 \le p \le 1$ and $Z_{p/2} = N\left(1 - p/2\right)$.

- The $100\left(1 - p\right)\%$ –confidence interval for R_t is given by $\left(m - z_{p/2} v,\ \ m + z_{p/2} v\right)$.
- The $100\left(1 - p\right)\%$ –prediction interval for S_t is given by $\left(S_0 e^{m - z_{p/2} v},\ \ S_0 e^{m + z_{p/2} v}\right)$.

It should be noted that the confidence interval for R_t is symmetric about the $\text{Med}\left[R_t\right] = m$, but the prediction interval for S_t is NOT symmetric about $\text{Med}\left[S_t\right] = S_0 e^m$.

Example 3.3

Assume the prices for a stock are modeled follow a lognormal model. You are given that $E\left[S_4\right] = 160$ and $\ln\left[\dfrac{E\left[S_4\right]}{\text{Med}\left[S_4\right]}\right] = 0.18$. Find the 90% prediction interval for S_4.

Example 3.4

The current price for a stock is 50. The stock pays continuous dividends at a rate of 3% and has an expected annual yield of 15%. The volatility of the stock is 40%. Assume the stock follows the lognormal model. The $100\beta\%$ -prediction interval for S_2 is $\left(26.2343,\ K\right)$. Find K and β.

Example 3.5

The price of a stock follows a lognormal model. The stock is currently worth 100, pays dividends at a rate of 2%, and a volatility of 30%. The 95%-prediction interval for the price of the stock after three years is $\left(K,\ 307.54\right)$. Find the expected annual yield for the stock.

3.3 ESTIMATING EXPECTED YIELD AND VOLATILITY

We saw in Section 2.4 that we can estimate a stock's volatility calculating the sample standard deviation of the yield rates for the stock over several consecutive periods. We called this estimate "historical volatility". In this section, we will take a similar approach to estimate not only the stock's volatility σ, but the stock's expected annual yield α. The process is described below.

- Let $S_0, S_1, S_2, \ldots, S_n$ be the prices of a stock observed at h-year intervals over a period of $n \cdot h$ years.
- Let $r_i = \ln(S_i / S_{i-1})$ be the observed periodic return during the ith period.
- We will assume that the periodic returns are independent and identically distributed. In other words, we assume that the r_i 's are independent observations of a random variable R_h.
- Let $\hat{m}$ and $\hat{v}$ be the sample mean and sample standard deviation of the r_i values. So, $\hat{m} = \left(\sum r_i\right) / n$ and $\hat{v} = \left[\sum (r_i - \hat{m})^2\right] / (n-1)$.
- We use formulas from Section 3.2 to obtain estimates for α and σ. These estimates are given by $\hat{\sigma} = \dfrac{\hat{v}}{\sqrt{h}}$ and $\hat{\alpha} = \dfrac{\hat{m}}{h} + \delta + 0.5\hat{\sigma}^2$.

We summarize these results below.

Estimating Expected Yield and Volatility

- Let $S_0, S_1, S_2, \ldots, S_n$ be the prices of a stock observed at h-year intervals.
- Calculate $r_i = \ln(S_i / S_{i-1})$. Then find $\hat{m} = \dfrac{1}{n}\sum r_i$ and $\hat{v} = \dfrac{1}{n-1}\sum (r_i - \hat{m})^2$.
- Our estimates are given by $\hat{\sigma} = \dfrac{\hat{v}}{\sqrt{h}}$ and $\hat{\alpha} = \dfrac{\hat{m}}{h} + \delta + 0.5\hat{\sigma}^2$.

As mentioned in Section 2.4, we can use the **1-Var Stats** calculator function to quickly find $\hat{m}$ and $\hat{v}$.

Example 3.6

A stock pays dividends continuously at a rate of 3%. Assume the stock follows a lognormal model. Prices for the stock on several different dates are provided below.

Date	Jan 1, 15	Mar 1, 15	May 1, 15	Jul 1, 15	Sep 1, 15	Nov 1, 15	Jan 1, 16
Price	63.58	71.82	65.19	56.37	68.63	77.94	66.57

a) Estimate the expected annual return and the volatility for the stock.

b) Find the expected price of the stock at the end of 2016.

Example 3.7

A stock pays dividends continuously at a rate of 2%. Assume the stock follows a lognormal model. Prices for the stock on several different dates are provided below.

Date	Jan 1, 15	Apr 1, 15	Jul 1, 15	Oct 1, 15	Jan 1, 16
Price	K	117	131	154	142

Based on this information, the historical volatility of the stock is 25.15%. Estimate the expected annual return for the stock.

Example 3.8

A stock pays dividends continuously at a rate of 1%. Assume the stock follows a lognormal model. Prices for the stock on several different dates are provided below.

Date	Jan 1, 15	Feb 1, 15	Mar 1, 15	Apr 1, 15	May 1, 15	Jun 1, 15	Jul 1, 15
Price	52	61	X	Y	X	59	54

Based on this information, the historical volatility of the stock is 39.68%. Estimate the expected annual return for the stock.

Example 3.9

A stock pays dividends continuously at a rate of 3%. Assume the stock follows a lognormal model. Prices for the stock on several different dates are provided below.

Date	Jul 1, 15	Aug 1, 15	Sep 1, 15	Oct 1, 15	Nov 1, 15	Dec 1, 15	Jan 1, 16
Price	X	86	92	81	74	84	X

Based on this information, the expected annual return is estimated to be 9.48%. Estimate the volatility of the stock.

3.4 EXPECTED OPTION PAYOFF

In order to price options on a stock, we need to be able to calculate the expected payoff of the option. To perform that task, we much first be able to calculate the probability that the option is exercised, as well as the price of the stock given that the option is exercised. These objectives will be the subject of this section.

Probability of Exercise

Consider a t-year, K-strike European call option on a stock. The call will only be exercised if $S_t > K$. The probability that the call will be exercised is thus equal to $Pr\left[S_t > K\right]$. Notice that:

$$Pr\left[S_t > K\right] = Pr\left[S_0 e^{R_t} > K\right] = Pr\left[e^{R_t} > K / S_0\right] = Pr\left[R_t > \ln\left(K / S_0\right)\right]$$

Assuming the lognormal model, the random variable R_t is normally distributed with mean $m = \left(\alpha - \delta - 0.5\sigma^2\right)t$ and standard deviation $v = \sigma\sqrt{t}$. Continuing the derivation above, we get:

$$Pr\left[S_t > K\right] = Pr\left[Z > \frac{\ln\left(K / S_0\right) - \left(\alpha - \delta - 0.5\sigma^2\right)t}{\sigma\sqrt{t}}\right] = 1 - N\left(\frac{\ln\left(K / S_0\right) - \left(\alpha - \delta - 0.5\sigma^2\right)t}{\sigma\sqrt{t}}\right)$$

We now use the fact that $1 - N(x) = N(-x)$ along with a little algebraic manipulation to obtain:

$$Pr\left[S_t > K\right] = N\left(-\frac{\ln\left(K / S_0\right) - \left(\alpha - \delta - 0.5\sigma^2\right)t}{\sigma\sqrt{t}}\right) = N\left(\frac{\ln\left(S_0 / K\right) + \left(\alpha - \delta - 0.5\sigma^2\right)t}{\sigma\sqrt{t}}\right)$$

For simplicity of notation, let $\hat{d}_2$ denote the quantity inside the standard normal CDF in the final expression. Thus, we have shown that $Pr\left[S_t > K\right] = N\left(\hat{d}_2\right)$. A very similar argument shows that $Pr\left[S_t < K\right] = N\left(-\hat{d}_2\right)$.

Probability of Exercise of European Options

- Let $\hat{d}_2 = \dfrac{\ln\left(S_0 / K\right) + \left(\alpha - \delta - 0.5\sigma^2\right)t}{\sigma\sqrt{t}}$.
- The probability of a t-year, K-strike European call being exercised is $Pr\left[S_t > K\right] = N\left(\hat{d}_2\right)$.
- The probability of a t-year, K-strike European put being exercised is $Pr\left[S_t < K\right] = N\left(-\hat{d}_2\right)$.

Example 3.10

Assume the prices for a stock follow the lognormal model. The current price of the stock is 120. The stock pays continuous dividends at a rate of 2%, has an expected annual return of 16%, and has a volatility of 40%. The continuously compounded risk-free rate is 4%. Find the probability that the following options on this stock will be exercised.

a) A two-year, 150-strike European call.

b) A two-year, 150-strike European put.

c) A four-year, at-the-money European call.

d) A four-year, at-the-money European put.

Example 3.11

Assume that the prices for a stock follow a lognormal model. You are given the following:

- $S_0 = 70$
- $\delta = 3\%$
- $\text{Var}\left[\ln\left(F_{0,3}^P\right)\right] = 0.12$

The probability that a three-year, 80-strike call on the stock will be exercised is 80%. Find the continuously compounded expected return on the stock.

Partial Expectation

Let X be a continuous random variable with density function given by $f(x)$. The **partial expectation of *X*, given that $\boldsymbol{X < K}$**, is denoted by $PE[X \mid X < K]$, and defined to be $PE[X \mid X < K] = \int_{-\infty}^{K} x\, f(x)\, dx$. This partial expectation can be thought of as the portion of the expected value that the values of X less than K contribute to. We will be interested in partial expectations for their use in the calculation of conditional expectations

We will now provide formulas for the partial expectations of lognormal random variables. The derivation of these formulas will be omitted. Let $X \sim \text{Normal}\left(m, v^2\right)$ and $Y = e^X$. Then:

$$PE[Y \mid Y < K] = E[Y]\, N\left(-\frac{m + v^2 - \ln K}{v}\right) \text{ and } PE[Y \mid Y > K] = E[Y]\, N\left(\frac{m + v^2 - \ln K}{v}\right)$$

To apply these formulas to the stock model yields the following:

$$PE[S_t \mid S_t < K] = S_0 e^{(\alpha-\delta)t} N\left(-\frac{\ln\left(S_0 / K\right) + \left(\alpha - \delta + 0.5\,\sigma^2\right)t}{\sigma\sqrt{t}}\right)$$

$$PE[S_t \mid S_t > K] = S_0 e^{(\alpha-\delta)t} N\left(\frac{\ln\left(S_0 / K\right) + \left(\hat{\alpha} - \delta + 0.5\,\sigma^2\right)t}{\sigma\sqrt{t}}\right)$$

To simplify the notation, we will let $\hat{d}_1 = \dfrac{\ln\left(S_0 / K\right) + \left(\alpha - \delta + 0.5\,\sigma^2\right)t}{\sigma\sqrt{t}}$. This gives us:

$$PE[S_t \mid S_t < K] = S_0 e^{(\alpha-\delta)t} N\left(-\hat{d}_1\right) \quad \text{and} \quad PE[S_t \mid S_t > K] = S_0 e^{(\alpha-\delta)t} N\left(\hat{d}_1\right)$$

Note that $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{t}$.

Partial Expectation

- Let $\hat{d}_1 = \dfrac{\ln\left(S_0 / K\right) + \left(\alpha - \delta + 0.5\,\sigma^2\right)t}{\sigma\sqrt{t}}$ and $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{t}$.
- $PE[S_t \mid S_t < K] = S_0 e^{(\alpha-\delta)t} N\left(-\hat{d}_1\right)$ and $PE[S_t \mid S_t > K] = S_0 e^{(\alpha-\delta)t} N\left(\hat{d}_1\right)$

Conditional Expectation

We now turn our attention to developing formulas for calculating the expected value of a stock *given* that the stock price is either greater or less than a certain threshold. In other words, we want to calculate the conditional expectation of a stock, given that a certain type of option was exercised.

In general, the conditional expectations $E[X \mid X < K]$ and $E[X \mid X > K]$ of a random variable are calculate using the formulas:

$$E[X \mid X < K] = \frac{PE[X \mid X < K]}{Pr[X < K]} \quad \text{and} \quad E[X \mid X > K] = \frac{PE[X \mid X > K]}{Pr[X > K]}.$$

Combining results from earlier in the section, we obtain the following formulas for conditional expectations for prices of stocks following the lognormal model:

Conditional Expectation

- Let $\hat{d}_1 = \dfrac{\ln(S_0 / K) + (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}$ and $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{t}$.
- $E[S_t \mid S_t < K] = \dfrac{S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)}{N(-\hat{d}_2)}$ and $E[S_t \mid S_t > K] = \dfrac{S_0 e^{(\alpha-\delta)t} N(\hat{d}_1)}{N(\hat{d}_2)}$

Example 3.12

Assume that the prices for a stock follow a lognormal model. You are given the following:

- $S_0 = 90$
- $\sigma = 30\%$
- $\alpha = 13\%$
- $\delta = 3\%$

a) Find $E[S_4 \mid S_4 < 100]$.

b) Find $E[S_4 \mid S_4 > 100]$.

c) Find $E[S_4]$.

Example 3.13

Assume prices for a stock follow the lognormal model. You are given the following:

- $\ln\left[E[S_3] / \text{Med}[S_3]\right] = 0.125$
- There is a 75.3063% chance that the price of the stock three years from now will be higher than the current price of the stock.
- If the stock price is higher three years from now, then the expected price will be 105.55.

Find the current price of the stock.

Example 3.14

The current price of a nondividend-paying stock is S_0. The continuously compounded expected annual yield for the stock is 8%. The continuously compounded risk-free rate of interest is 3%. According to the lognormal model, the probability that the time-*T* price of the stock is greater than $0.8S_0$ is 0.6586. The expected price of the stock at time *T*, given that the price is greater than $0.8S_0$, is $1.3128F_{0,T}$. Find the volatility of the stock.

Expected Option Payoff

We now have all of the necessary tools for calculating the expected payoff of European options. For any given European option, the expected payoff will be equal to the probability that the call is exercised times the expected payoff of the option given that it is exercised. We will start by considering the situation for calls.

Consider a t-year, K-strike European call on a stock whose prices follow the lognormal model. We can calculated the expected payoff of the call as follows:

$$\begin{aligned}
E[\text{Call PO}] &= \left(Pr[\text{Call is exercised}]\right)\left(E\left[PO \text{ of call} \mid \text{Call is exercised}\right]\right) \\
&= \left(Pr[S_t > K]\right)\left(E[S_t - K \mid S_t > K]\right) \\
&= N(\hat{d}_2)\left(E[S_t \mid S_t > K] - K\right) \\
&= N(\hat{d}_2)\left(\frac{S_0 e^{(\alpha-\delta)t} N(\hat{d}_1)}{N(\hat{d}_2)} - K\right) \\
&= S_0 e^{(\alpha-\delta)t} N(\hat{d}_1) - K\,N(\hat{d}_2)
\end{aligned}$$

A similar argument can be used to show that $E[\text{Put PO}] = K\,N(-\hat{d}_2) - S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$. In summary:

Expected Option Payoff

- $E[\text{Call PO}] = S_0 e^{(\alpha-\delta)t} N(\hat{d}_1) - K\,N(\hat{d}_2)$ and $E[\text{Put PO}] = K\,N(-\hat{d}_2) - S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$

Example 3.15

Let S_t be the time-t price of a stock paying dividends at a continuously compounded rate of 2%. The current price of the stock is 50, and $\ln(S_2 / S_0) \sim \text{Normal}(m = 0.08,\ v^2 = 0.32)$. Find the expected payoff of an two-year 60-strike European call on this stock.

Example 3.16

Assume prices for a nondividend-paying stock follow the lognormal model. The stock is currently worth 80 and an expected annual yield of 6%. You are also given that $\ln\left[E[S_4] / \text{Med}[S_4]\right] = 0.32$. Given that $\hat{d}_2 = 0.44607$, find the expected payoff of a four-year European put on this stock.

Example 3.17

You are given the following information about the time T price of a stock:

- The partial expectation of the price, given that it is less that than K, is 31.1647.
- The conditional expectation of the price, given that it is less than K, is 54.2328.
- $\text{Var}\left[\ln(S_T)\right] = 0.48$.

Find the expected payoff of a T-year K-strike put on the stock.

Since we do not know the yield rates on the options we are not able to discount the expected payoffs to time 0 in order to find the option premiums. This is similar to the issue we encountered with binomial trees when using true probabilities. In the next section, we will learn how to apply a risk-neutral version of the above analysis to price these options. That method will result in the what is called the Black-Scholes formula.

3.5 BLACK-SCHOLES FORMULA FOR STOCK OPTIONS

Derivation of Black-Scholes Formula

In the previous section we learned how to calculated the expected payoff of a European call or put on a stock whose price follows the lognormal stock model. If we knew γ, the expected yield of the option, we could use that rate to discount the expected payoff to time 0 to find the premium. We do not typically know γ prior to knowing the option premium, however.

This situation should be familiar. We encountered the same difficulty when using binomial trees to price options. The solution in that case was to switch from true probabilities to risk-neutral probabilities. We will use the same approach here.

We begin by defining risk-neutral versions of $\hat{d}_1$ and $\hat{d}_2$, which we will call d_1 and d_2. This is done by simply replacing the yield rate α with the risk-free rate r. This yields the following expressions:

$$d_1 = \frac{\ln(S_0 / K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S_0 / K) + (r - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Recall from the previous section that the expected payoffs for a European call and put are given by:

$$E[\text{Call PO}] = S_0 e^{(\alpha-\delta)t} N(\hat{d}_1) - K\,N(\hat{d}_2) \quad \text{and} \quad E[\text{Put PO}] = K\,N(-\hat{d}_2) - S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$$

Risk-neutral versions of these quantities are given by:

$$E^*[\text{Call PO}] = S_0 e^{(r-\delta)t} N(d_1) - K\,N(d_2) \quad \text{and} \quad E^*[\text{Put PO}] = K\,N(-d_2) - S_0 e^{(r-\delta)t} N(-d_1)$$

We discount these expressions to time 0 using the risk-free rate to find the option premiums:

$$\text{Call} = S_0 e^{-\delta t} N(d_1) - K e^{-rt} N(d_2) \quad \text{and} \quad \text{Put} = K e^{-rt} N(-d_2) - S_0 e^{-\delta t} N(-d_1)$$

These formulas are called the **Black-Scholes option pricing formulas**.

Black-Scholes Formula for Stock Options

- Let $d_1 = \frac{\ln(S_0/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln(S_0/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$.
- The price of a European call on a stock is given by $\text{Call} = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)$.
- The price of a European put on a stock is given by $\text{Put} = K e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$.

The formulas above apply when the stock pays continuous (or no) dividends. If the stock pays discrete dividends, you can price the option using a general formula discussed later in Section 3.7. Also keep in mind that the Black-Scholes formula is used to price European options, and is not valid for American options.

Example 3.18

Assume the Black-Scholes framework. The current price of a stock is 90. The stock pays dividends continuously at a rate of 1% and has a volatility of 30%. The risk-free rate is 4%.

a) Calculate the premium for a two-year 90-strike call on the stock.

b) Calculate the premium for a two-year 90-strike put on the stock.

Example 3.19

Assume the Black-Scholes framework. The current price of a stock is 40. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate is 6%. You are given the following information regarding a six-month European put on the stock: $d_1 = -0.464756$ and $d_2 = -0.641533$. Find the price of this put.

3.6 BLACK-SCHOLES FORMULA FOR CURRENCY OPTIONS

Although the most common application for the Black-Scholes formula is pricing options on stocks, this tool can be used to price options on many different types of assets. In this section, we will see how to use the Black-Scholes formulas to price options on currency exchanges.

Notation and Terminology for Currency Exchanges

Assume that we are interested in converting between two currencies, one domestic and one foreign, at some point in the future. Options may be used to provide insurance by placing caps on the exchange rate.

- For convenience, we will denote the domestic currency with $\$_d$ and the foreign currency with $\$_f$.
- Assume r_d is the risk-free rate for domestic currency and r_f is the risk-free rate on the foreign currency.
- Assume one unit of the foreign currency currently costs x_0 units of the domestic currency. That is, $\$_f 1 = \$_d x_0$.
- A *K*-strike call on the foreign currency gives the right to purchase $\$_f 1$ for a price of $\$_d K$.
- A *K*-strike put on the foreign currency gives the right to sell $\$_f 1$ for a price of $\$_d K$.
- The "denominating" currency for a currency exchange options represents to domestic currency, or in other words, the currency that the strike price and premium are paid in. For instance, a "Euro-denominated call on a dollar" is an option that allows for the purchase of a dollar, and will have the strike price and option premium paid in Euros.

Pricing Options on Currency Exchanges

Options on currency exchanges can be priced using the standard Black-Scholes formula by making a few substitutions. The process is described below.

Black-Scholes Formula for Currency Exchange Options

- Make the following substitutions: $r = r_d$, $\delta = r_f$, and $S_0 = x_0$.
- Let σ be the volatility in the exchange rate.
- Then $d_1 = \dfrac{\ln\left(x_0 / K\right) + \left(r_d - r_f + 0.5\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.
- The price of a European call on the exchange is $Call = x_0 e^{-r_f T} N\left(d_1\right) - K e^{-r_d T} N\left(d_2\right)$.
- The price of a European put on the exchange is $Put = K e^{-r_d T} N\left(-d_2\right) - x_0 e^{-r_f T} N\left(-d_1\right)$.

Example 3.20

Assume the Black-Scholes framework. The current yen-to-dollar exchange rate is 110 yen per dollar. The annual volatility of the exchange is 30%. The continuously compounded risk-free rate for yen is 3% and the continuously compounded risk-free rate for dollars is 5%. Calculate the premium for a 9-month, 100-strike, yen-denominated European put on one dollar.

Example 3.21

A financial institution knows that it will need to purchase €200,000 two years from now in order to satisfy certain obligations. The company decides to hedge against rising exchange rates and purchases options allowing them to pay no more than $250,000 for the euros. Find the price of these options, in dollars, given the following information:

- The current exchange rate is €0.8 per $1.
- The continuously compounded risk-free rate for dollars is 4%.
- The continuously compounded risk-free rate for euros is 3%.
- The volatility of the exchange rate is 20%.

Option Duality

Recall that any call can be viewed as a put by switching the underlying asset and the strike asset, and vice-versa. This concept is known as option duality, and was discussed in Section 1.3. It can be summarized using the following equation: $C(A=Y,\ K=X) = P(A=X,\ K=Y)$. Option duality is particularly useful when working with options on currency exchanges.

3.7 GENERAL BLACK-SCHOLES FORMULA

Before moving on to discussed other specific applications of the Black-Scholes formula, it will be useful to introduce a more general version of the formula than the one introduced in Section 3.5. This new formula can be used to price options on a wide variety of assets, such as stocks, bonds, and futures.

Assume an asset (not necessarily a stock) currently has a value of S. Consider a European option on the asset expiring at time T with strike price K. We can use the formulas below to price such an option.

General Black-Scholes Formula

- Let $F^P(S)$ be the prepaid forward price of the asset.
- Let $F^P(K)$ be the prepaid forward price of a payment of K at time T.
- Let $d_1 = \dfrac{\ln\left(F^P(S)/F^P(K)\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$ and $d_2 = \dfrac{\ln\left(F^P(S)/F^P(K)\right) - 0.5\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$.
- The price of a European call on the asset is given by $\text{Call} = F^P(S)N(d_1) - F^P(K)N(d_2)$.
- The price of a European put on the asset is given by $\text{Put} = F^P(K)N(-d_2) - F^P(S)N(-d_1)$.

The values of $F^P(S)$ and $F^P(K)$ will depend on the type of assets being used for the option. However, if the strike asset is a fixed amount of cash, then we will always set $F^P(K) = K e^{-rT}$.

Alternate Formula for d_1

Prepaid forward prices are always the PV of forward prices, and thus $F^P(S) = F(S)e^{-rT}$ and $F^P(K) = F(K)e^{-rT}$. It follows that $F^P(S)/F^P(K) = F(S)/F(K)$, and thus $d_1 = \dfrac{\ln\left(F(S)/F(K)\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$. If the strike asset is a fixed amount of cash, then $F(K) = K$ and $d_1 = \dfrac{\ln\left(F(S)/K\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$. This provides us with two equivalent formulas for d_1. Each will have their advantages in different situations.

One application of the general Black-Scholes formula is to price options on stocks paying discrete dividends. This is illustrated in the next example.

Example 3.22

Assume the Black-Scholes framework applies. The current price of a stock is 100. The stock is scheduled to pay a dividend of 5 in nine months. The continuously compounded risk-free rate of interest is 8%, and $\text{Var}\left[\ln\left(F^P_{0,1}\right)\right] = 0.04$. Find the price of a one-year 95-strike European put on this stock.

3.8 BLACK-SCHOLES FORMULA FOR OPTIONS ON FUTURES

In this section, we will apply the general Black-Scholes formula to price options on futures contracts. Recall that a futures contract is a financial instrument that locks in the price of an asset that is to be delivered at some later date. The underlying asset for a future is often a stock, an index, or a physical commodity such as oil, gold, or wheat. To simplify the development of our pricing formulas, we will assume that the options considered are on assets that do not pay dividends. However, the formulas we obtain are valid even if the asset underlying the futures contract does pay dividends.

Consider a futures contract on a commodity, and an option on that futures contract. Assume that the option expires at time T and the futures contract expires at time $T+S$. Let P be the value of the commodity today. The price of the futures contract, to be paid at time $T+S$, is $F_{0,T+S} = Pe^{r(T+S)}$. If we wanted to pay for the commodity prior to delivery, say at time T instead of at time $T+S$, the appropriate price would be $Pe^{r(T+S)}$ discounted back to time T using the risk-free rate. This would give Pe^{rT}. Notice that the quantity Pe^{rT} is also equal to the price for a futures contract that would expire at time T, when the option expires. This is the price that should be compared to the strike price, which is what will actually be paid for the contract at time T if the option is exercised.

The discussion in the previous paragraph indicates that the actual date of expiration for the futures contract does not have any influence on the price of the option. Regardless of when the futures contract expires, we will always use the price of a futures contract that expires at the same time as the option we are pricing.

A summary of the formulas involved in pricing options on futures is provided below.

Black-Scholes Formula for Options on Futures

- Let F be the price for a futures contract expiring at time T.
- Let K be the strike price for a European option on the futures contract.
- Let $d_1 = \dfrac{\ln(F/K) + 0.5\sigma^2}{\sigma\sqrt{T}}$ and $d_2 = \dfrac{\ln(F/K) - 0.5\sigma^2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$.
- The price of a European call on the futures contract is $\text{Call} = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2)$.
- The price of a European put on the futures contract is $\text{Put} = Ke^{-rT}N(-d_2) - Fe^{-rT}N(-d_1)$.

Example 3.23

Assume the Black-Scholes framework applies. The one-year futures price for a certain commodity is currently 40. The price of a one-year at-the-money European call on the futures contract is 5.33893. The continuously risk-free rate of interest is 4%. Find the price of a 35-strike one-year European call on the futures contract.

Example 3.24

Assume the Black-Scholes framework applies. The one-year futures price for a certain commodity is currently 70. The price of a one-year at-the-money European put on the futures contract is 7.86042. The continuously risk-free rate of interest is 6%. Three months later the nine-month futures price is 75. Find the value of the put at that time.

Example 3.25

Assume the Black-Scholes framework applies. You are given the following information: A four-year at-the-money European call on a futures contract for Commodity A currently costs 15 more than a four-year at-the-money European call on a futures contract for Commodity B. The volatility of futures prices for both commodities is 20%. Find the absolute difference in the prepaid forward prices of the two commodities.

3.9 BLACK-SCHOLES FORMULA FOR BOND OPTIONS

We will now look at how to price an option on a zero-coupon bond. We will apply the general Black-Scholes formula to develop our pricing formulas, but we first need to introduce some notation regarding the prices of zero-coupon bonds.

Bond Price Notation

The notation introduced below is convenient for working with zero-coupon bonds. We will use this notation frequently in Chapter 8 when we study interest rate models.

- $P(t, T)$ denotes the price, set at time *t*, of a bond to be purchased at time *t*, maturing for $1 at time *T*.
- $P_0(t, T)$ denotes the price, set at time 0, of a bond to be purchased at time *t*, maturing for $1 at time *T*.
- Notice that $P_0(t, T)$ is the forward price of $P(t, T)$. As such, we also denote $P_0(t, T)$ by $F_{0,T}(t, T)$.
- Interest theory concepts tell us that $P_0(t, T) = \dfrac{P(0,T)}{P(0,t)}$.
- Notice that if risk free rate is constant, $P(0,T) = e^{-rT}$. Even if the risk free rate varies, we can use $P(0,T)$ as a present value factor. If we want the present value of a payment occurring at time *T*, we multiply by $P(0,T)$.

Pricing Options on Bonds

The formulas used in pricing an option on a bond are provided below.

Black-Scholes Formula for Options on Bonds

- Consider a zero-coupon bond maturing for $1 at time $T + S$.
- An option allows for the purchase or sale of the bond at time *T* for a strike price of *K*.
- The time *T* forward price of the bond is $F = F_{0,T}(t, T) = P_0(T,\ T + S) = \dfrac{P(0,\ T + S)}{P(0,\ T)}$.
- Let $d_1 = \dfrac{\ln(F / K) + 0.5\sigma^2}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.
- The price of a European call on the bond is given by $C = P(0,\ T)\left[F\,N(d_1) - K\,N(d_2)\right]$.
- The price of a European put on the bond is given by $P = P(0,\ T)\left[K\,N(-d_2) - F\,N(-d_1)\right]$.

Additional Exercises

Section 3.1

1.

Section 3.2

2. Assume that prices for a stock follow the lognormal model. The current price of the stock is 60. The stock pays dividends continuously at a rate of δ. You are given that $E\left[S_4\right] = 86$ and $E\left[\ln\left(S_4\right)\right] = 4.32934$. Find the volatility of the stock.

3. The current price for a stock is 120. The stock pays continuous dividends at a rate of 4% and has an expected annual yield of 16%. The volatility of the stock is 35%. Assume the stock follows the lognormal model. There is a 90% chance that the price of the stock in 3 years will be greater than *K*. Find *K*.

CHAPTER 4 – The Greeks

4.1 THE GREEKS

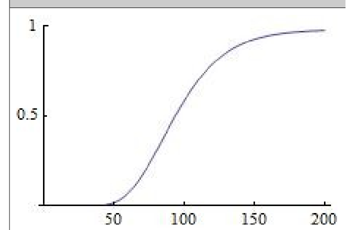
Let V represent the value of an option that has been priced using the Black-Scholes formula. Then V is a continuous function several variables, such as the stock price S, the strike price K, the asset volatility σ, the risk-free rate r, and the dividend rate δ. We can differentiate V with respect to any of these variables. **The Greeks** refer to a specific set of partial derivatives of V.

Delta

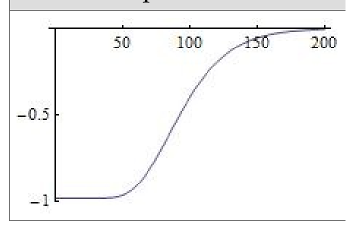
Delta, denoted by Δ, is the partial derivative of V with respect to the stock price S. That is to say that $\Delta = V_S$. Delta is perhaps the most widely used of the Greeks. As we will see later, it is closely related to the value of the same name that we studied for binomial trees. Several facts relating to delta are provided below.

- The value of a call increases as the stock price increases. However, the increase in the value of the call is always less than or equal to the increase in the stock price. Thus, $0 \leq \Delta_C \leq 1$.
- The value of a put decreases as the stock price increases. However, the decrease in the value of the put is always less than or equal to the increase in the stock price. Thus, $-1 \leq \Delta_P \leq 0$.
- It can be shown that, under the Black-Scholes framework, the delta for a call is given by $\Delta_C = e^{-\delta T} N(d_1)$ and the delta for a put is given by $\Delta_P = -e^{-\delta T} N(-d_1)$. The derivations of these formulas are omitted.
- Differentiating both sides of the put-call parity equation with respect to S yields $\Delta_C - \Delta_P = e^{-\delta T}$.
- Graphs of Δ as a function of S are shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Graph of Δ_C



Graph of Δ_P



Example 4.1

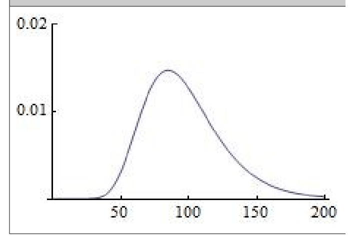
Assume the Black-Scholes framework applies. The continuously compounded risk-free rate of interest is 6%. A two-year at-the-money European call on the stock currently has a delta of 0.638265. An otherwise similar put has a delta of -0.322524. Determine the volatility of the stock.

Gamma

Gamma, denoted by Γ, is the second partial derivative of V with respect to the stock price S. That is to say that $\Gamma = V_{SS}$. Several facts relating to gamma are provided below.

- Gamma for a call is always equal to gamma for a put with the same parameters. That is, $\Gamma_C = \Gamma_P$.
- Gamma is always greater than or equal to zero.
- Gamma is given by the formula $\Gamma = \dfrac{1}{S\sigma\sqrt{2\pi T}} e^{-0.5\delta T d_1^2}$.
- A graph of Γ as a function of S is shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Graph of Γ

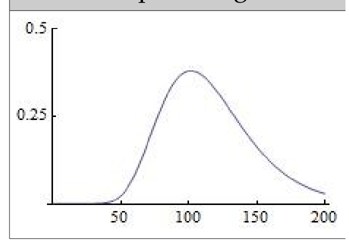


Vega

Vega is the partial derivative of V with respect to the volatility σ. That is to say that $vega = V_\sigma$. Several facts relating to vega are provided below.

- Vega for a call is always equal to vega for a put with the same parameters. That is, $vega_C = vega_P$.
- Vega is always greater than or equal to zero.
- Vega is occasionally defined as $vega = 0.01\,V_\sigma$. This definition of vega results from measuring σ as a percentage.
- A graph of vega as a function of S is shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Graph of vega

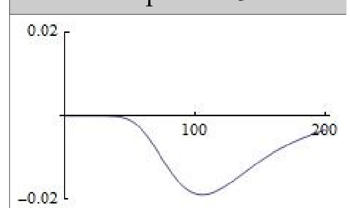


Theta

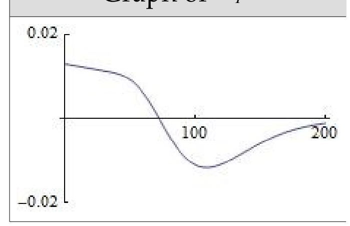
Theta, denoted by θ, is the partial derivative of V with respect to *t*, the time elapsed since the option was purchased. That is, $\theta = V_t$.

- By differentiating the expression for put-call parity, one can obtain the following relationship between theta for a call and theta for a put: $\theta_C - \theta_P = \delta S\, e^{-\delta(T-t)} - r\,K\,e^{-r(T-t)}$.
- Theta is occasionally defined to be $\theta = V_t / 365$. This definition of theta results from measuring *t* in days rather than years.
- Graphs of θ as a function of S are shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Graph of θ_C



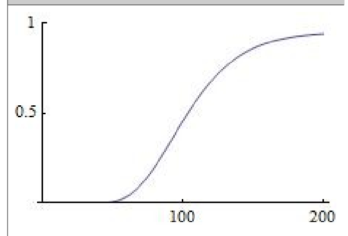
Graph of θ_P



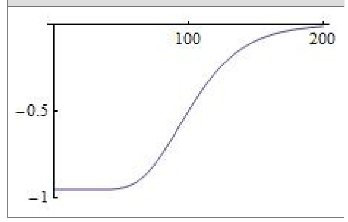
Rho

Rho, denoted by ρ, is the partial derivative of V with respect to the risk-free rate *r*. That is, $\rho = V_r$.

Graph of ρ_C



Graph of ρ_P

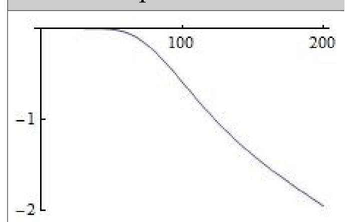


- By differentiating the expression for put-call parity, one can obtain the following relationship between rho for a call and rho for a put: $\rho_C - \rho_P = T\,K\,e^{-rT}$.
- Rho for a call is always greater than or equal to zero. Rho for a put is always less than or equal to zero. These facts can be easily recalled by noting to role of *r* in the expressions $Call = x_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2)$ and $Put = K e^{-r_d T} N(-d_2) - x_0 e^{-r_f T} N(-d_1)$.
- Rho is occasionally defined to be $\rho = 0.01 V_r$. This definition of ρ results from measuring σ as a percentage.
- Graphs of ρ as a function of S are shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

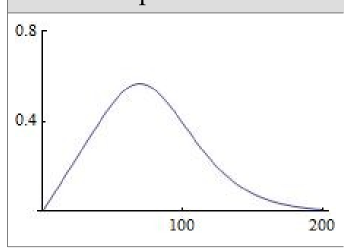
Psi

Psi, denoted by Ψ, is the partial derivative of V with respect to the dividend rate δ. That is, $\rho = V_\delta$.

Graph of Ψ_C

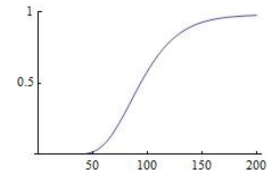
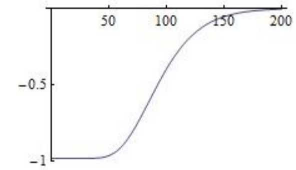
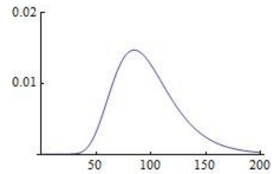
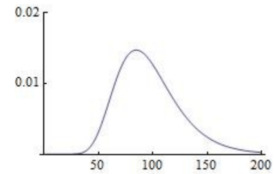
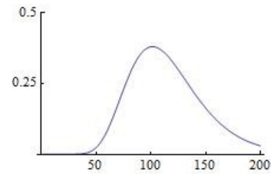
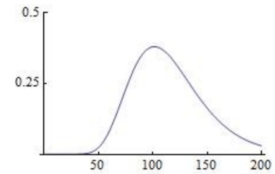
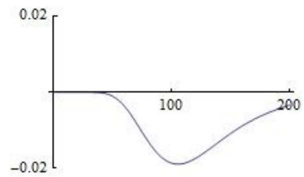
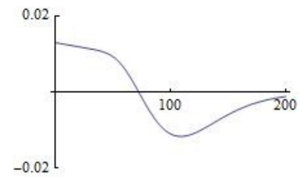
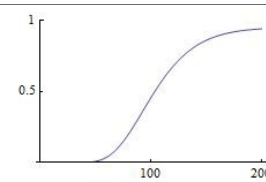
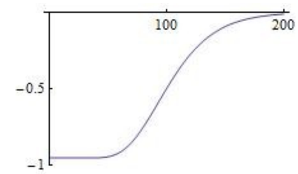
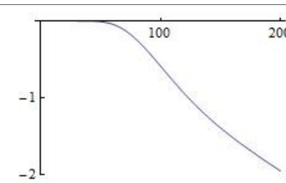
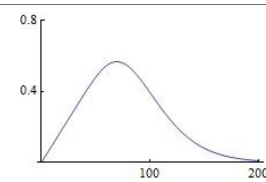


Graph of Ψ_P



- By differentiating the expression for put-call parity, one can obtain the following relationship between psi for a call and psi for a put: $\psi_C - \psi_P = -T\,S\,e^{-\delta T}$.
- Psi for a call is always less than or equal to zero. Psi for a put is always greater than or equal to zero. These facts can be easily recalled by noting to role of δ in the expressions $Call = x_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2)$ and $Put = K e^{-r_d T} N(-d_2) - x_0 e^{-r_f T} N(-d_1)$.
- Psi is occasionally defined to be $\Psi = 0.01 V_\delta$. This definition of Ψ results from measuring σ as a percentage.
- Graphs of Ψ as a function of S are shown on the right. The other parameters are held constant at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Summary of the Greeks

Greek	Derivative	Comments	Graph for Call	Graph for Put
Delta Δ	$\frac{\partial V}{\partial S}$	• $-1 \le \Delta_P \le 0 \le \Delta_C \le 1$ • $\Delta_C = e^{-\delta T} N(d_1)$ • $\Delta_P = -e^{-\delta T} N(-d_1)$ • $\Delta_C - \Delta_P = e^{-\delta T}$		
Gamma Γ	$\frac{\partial^2 V}{\partial S^2}$	• $\Gamma_C = \Gamma_P$ • $\Gamma \ge 0$ • $\Gamma = \frac{1}{S\sigma\sqrt{2\pi T}} e^{-0.5\delta T d_1^2}$		
Vega	$\frac{\partial V}{\partial \sigma}$	• $vega_C = vega_P$ • $vega \ge 0$ • Occasionally defined by $vega = 0.01\, V_\sigma$		
Theta θ	$\frac{\partial V}{\partial t}$	• The variable t represents time that has elapsed since purchase of the option. • $\theta_C - \theta_P = \delta S e^{-\delta(T-t)} - r K e^{-r(T-t)}$ • Occasionally defined by $\theta = V_t / 365$		
Rho ρ	$\frac{\partial V}{\partial r}$	• $\rho_C - \rho_P = T K e^{-rT}$ • $\rho_C \ge 0$ • $\rho_P \le 0$ • Occasionally defined by $\rho = 0.01 V_r$.		
Psi Ψ	$\frac{\partial V}{\partial \delta}$	• $\psi_C - \psi_P = -T S e^{-\delta T}$ • $\psi_C \le 0$ • $\psi_P \ge 0$ • Occasionally defined by $\Psi = 0.01 V_\delta$.		

Note: The horizontal axis in each of the graphs above represents the stock price, S. The values of the other parameters are set at $K = 100$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.02$, and $T = 1$.

Greeks for a Portfolio

Assume a portfolio consisting of n options. Assume the current value of those options to the owner of the portfolio are given by $V_1, V_2, \ldots, V_n$. The the current value of the portfolio is given by $\pi = V_1 + V_2 + \ldots + V_n$. Let G denote an arbitrary Greek. Assume that value of this Greek for the options in the portfolio are given by $G_1, G_2, \ldots, G_n$. The value of this Greek for the portfolio is given by:

- $$G_\pi = \frac{G_1 V_1 + G_2 V_2 + \ldots + G_n V_n}{V_1 + V_2 + \ldots + V_n} = \frac{G_1 V_1 + G_2 V_2 + \ldots + G_n V_n}{\pi}$$

In other words, the Greek for a portfolio is the weighted average of that Greek overall all of the assets in the portfolio, with the weights determined by the value of the assets.

4.2 DELTA HEDGING

Market Makers

As discussed in Section 2.2, a **market maker** is an entity that facilitates the buying and selling of stocks, as well as financial derivatives such as options. A market maker is generally not interested in making a profit by holding a position with respect to a stock. They instead make money by charging small commissions to any individual using their services to conduct a trade, whether that individual is buying an asset or selling an asset.

The ideal situation for a market maker is that whenever they sell an option, they are also able to buy an identical option. The market maker would collect commissions on both transactions and would maintain a neutral position with respect to the underlying option. When the number of options bought and sold are not equal, the market maker can offset their position by buying or selling shares of the underlying stock. The most common strategy for using shares of a stock to offset an undesired position in that stock is the method of delta hedging.

Delta Hedging

As described above, a market maker can buy or sell shares of a stock to offset a long or short position resulting from buying or selling an option on that stock. When employing the method of **delta hedging**, the number of shares that the market maker buys or sells will be equal to the delta of the option being offset.

If the market maker is attempting to offset a long position, they will sell shares of the stock. If the position being offset is short, then the market maker will purchase shares of the stock.

Example 4.2

Assume the Black-Scholes framework applies. The continuously compounded risk-free rate of interest is 5%. A share of ABC stock is currently worth 100. The stock pays continuous dividends at a rate of 2% and has a volatility of 30%. Determine the number of shares that must be bought or sold in order to delta hedge 1000 units of each of the following two-year European options on this stock.

a) A purchased 110-Strike Call.

b) A written 110-Strike Call.

c) A purchased 110-Strike Put.

d) A written 110-Strike Put.

To understand why delta hedging is an effective hedging strategy, assume that a portfolio consists of one option as well as n shares of the underlying stock. Then the value of the portfolio will be given by $\pi = V + nS$. We would like the derivative of π with respect to S to be 0, so that small changes in the stock price will result in small changes in the value of the portfolio as a whole. Differentiating both sides of $\pi = V + nS$ with respect to S yields $\pi_S = \Delta + n$. Setting $\pi_S = 0$ gives us $n = -\Delta$. This shows that the number of shares that we need to buy or sell is equal to Δ . If n turns out to be negative, then we should sell S shares to offset the position. If n is positive, then we should buy shares to offset the position.

Delta Hedging a Portfolio

Assume we wish to delta hedge a portfolio consisting of several options, all with the same underlying stock. The options could be call or puts, purchased or written, and could have the same or different strike prices and times until expiration. To delta hedge this portfolio, we simply add in some number of stocks so that the overall delta for the portfolio is equal to zero. Pay careful attention to the signs of various quantities to determine whether the shares should be purchased or sold.

Example 4.3

Assume the Black-Scholes framework applies. The continuously compounded risk-free rate of interest is 4%. A share of nondividend-paying stock is currently worth 120. The volatility of the stock is 20%. A portfolio consists of the following European options:

- 100 one-year, 130-strike written calls
- 60 one-year, 130-strike written puts
- 80 two-year at-the-money purchased calls

Determine the number of shares that must be bought or sold in order to delta hedge this portfolio.

Overnight Profit

We know the value of a delta hedged portfolio should change only slightly as a result of small changes in the stock price. It is likely, however, that the owner of such a portfolio will experience a small profit if the stock price changes. We will now discuss how to calculate the profit or loss for a delta hedged portfolio.

Assume that a portfolio of options on a certain stock currently has a value of V_0 and a delta of Δ.

- Denote the value of the unhedged portfolio at time *t* by V_t.
- Denote the value of the hedged portfolio at time *t* by $\pi_t = V_t - \Delta e^{\delta t} S_t$.
- The profit for the unhedged and hedged portfolios from $t = 0$ to $t = h$ are given by
  - Unhedged: $Profit = V_h - FV\left(V_0\right)$.
  - Hedged: $Profit = \pi_h - FV(\pi_0) = \left[V_h - \Delta e^{\delta h} S_h\right] - \left[V_0 - \Delta S_0\right] e^{rh}$
- It can be shown that the portfolio will have a positive profit if $S_0 - S_0\sigma\sqrt{h} < S_h < S_0 + S_0\sigma\sqrt{h}$.

Example 4.4

Assume the Black-Scholes framework applies. A market maker sells 200 one-year, at-the-money European call options on a nondividend-paying stock and delta hedges the position. The current price for one share of the stock is 100, and the volatility of the stock is 40%. The continuously compounded risk free rate is 5%.

a) Find the number of shares bought or sold in the hedging portfolio.
b) Find the overnight profit for the hedged and unhedged portfolio if the price of the stock after one day is 95, 100, or 105.
c) Find the stock prices that would result in the market-maker breaking even after the one day.

Example 4.5

The current price of a nondividend-paying stock is 120. A market maker creates a portfolio by longing 60 one-year 125-strike European calls and shorting 40 one-year 130 strike European calls. The market maker delta hedges the positions. You are given the following information regarding the options:

- The 125-strike calls each have a premium of 13.74 and a delta of 0.5454.
- The 130-strike calls each have a premium of 11.80 and a delta of 0.4933.

One day later, the stock price has increased to 122.

- The new premium for the 125-strike calls is 14.83.
- The new premium for the 130-strike calls is 12.78.

Calculate the market maker's overnight profit. Assume $r = 3\%$.

Example 4.6

An investor buys 100 units of a T-year European put on a nondividend-paying stock and delta-hedges the position. At this time, the price of the stock is 70, the premium for the put is 4.66, and the put delta is -0.2311.The premium for an equivalent call option is 18.15.

At time $t < 1$, the investor closes out the portfolio. When she closes the portfolio, the stock price is 60 and the put premium is 4.31. The premium for an equivalent call option is 5.20.

Determine the investor's profit or loss.

Rebalancing Portfolios

Note that Δ for a call or a put is a function of S, K, T, δ, r, and σ. The strike price of an option is certainly fixed, and we tend to work under the assumption that δ, r, and σ will remain constant over the lifetime of the option as well. However, the stock price S and the time until expiration T will certainly change over time. As a result, Δ will change over time a delta hedged portfolio must continually be rehedged (or rebalanced) by buying or selling shares of the stock.

Example 4.7

A market maker sells 200 6-month European puts on a stock that pays dividends continuously at a rate of 3%. The market maker immediately delta hedges the position, and then rehedges after one week. The continuously compounded risk-free rate of interest is 4%. You are given:

- At the time of the sale the stock price is 150, the premium for one unit of the put is 14.15, and delta for the put is -0.4362.
- One week after the sale, the stock price is 160, the premium for one unit of the put is 10.03, and the delta for the put is -0.3371.

a) Find the market maker's profit after one week.
b) Find the cost to rebalance the portfolio after one week.

Frequent rehedging will decrease the chance that the owner of the portfolio will see a large profit or loss. Assuming that the owner of the portfolio rehedges it every h years, the periodic and annual variances in the portfolio's return can be calculated using the **Boyle-Emanual formulas**. These formulas are given below.

- The periodic variance in portfolio's return is given by $\text{Var}\left[R_h\right] = 0.5\left(S^2\sigma^2\Gamma h\right)^2$.
- The annual variance in the portfolio's return is given by $\text{Var}\left[R\right] = 0.5\left(S^2\sigma^2\Gamma\right)^2 h$.

Hedging Multiple Greeks

Notice that delta hedging works by creating a linear approximation of the change in the option price as a function of the price of the underlying stock. This provides a good approximation for small changes in the stock price over short periods of time, but it does have its limitations. In order to develop a hedging strategy that better matches the changes in the value of our option, it would be a good idea to try try to incorporate information provided by Greeks other than just delta. For instance, if we wanted to create a quadratic approximation of the option price as a function of the stock price, we could consider using gamma in our hedging strategy. If we wanted to try to capture the effect that the passing of time had on the value of our option, we could utilize theta. It turns out that we can, in fact, hedge a portfolio on several Greeks at once. The process is detailed below.

- For each Greek being hedged on, you will need a separate asset in your hedging portfolio.
- Each Greek being hedged on should have a value of zero for the entire portfolio.
- Recall that the value of a Greek for a portfolio is the sum of that Greek over all assets in the portfolio.
- One instrument in the portfolio can always be the underlying stock.
- The underlying stock has a delta of $\Delta = 1$. All other Greeks for the stock are zero.

Example 4.8

You are given the following information regarding options on a stock:

- For a one-year 50-strike call on the stock, $\Delta = 0.8609$ and $\Gamma = 0.03252$.
- For a one-year 60-strike call on the stock, $\Delta = 0.5996$ and $\Gamma = 0.03320$.

A market maker sells 400 units of the 50-strike call. Determine the number of 60-strike calls and shares of the stock that need to be bought or sold in order to delta-gamma hedge this position.

Example 4.9

You are given the following information regarding options on a stock:

- For a two-year 120-strike call on the stock, $\Delta = 0.3888$ and $\Gamma = 0.014084$.
- For a two-year 120-strike put on the stock, $\Delta = -0.5530$.

A market maker sells 2000 units of the call option. Determine the number of put options and shares of the stock that need to be bought or sold in order to delta-gamma hedge this position.

Example 4.10

You are given the following information regarding options on a stock:

- For a 1-year 90-strike call on the stock, $\Delta = 0.7711$, $\Gamma = 0.02594$, and $\theta = -2.4$.
- For a 1-year 100-strike call on the stock, $\Delta = 0.5486$, $\Gamma = 0.02656$, and $\theta = -3.34$.
- For a 1-year 110-strike call on the stock, $\Delta = 0.3152$, $\Gamma = 0.02645$, and $\theta = -2.97$.

A market maker shorts 1000 units of the 100-strike call option. Determine the number of 90-strike call options, 110-strike call options, and shares of the stock that need to be bought or sold in order to delta-gamma-theta hedge this position.

4.3 DELTA-GAMMA-THETA APPROXIMATION

Taylor Series

Recall that the Taylor series for a function $f(x)$ centered at a point a is given by $\sum_{i=0}^{\infty} \frac{1}{n!} f^{(n)}(x-a)^n$. Taylor series are often used to create polynomial approximations for $f(x)$ for values of x sufficiently close to a. For example, by taking the first three terms of the Taylor Series, we obtain the second degree Taylor polynomial of $f(x)$ centered at a, which is given by $T_2(x) = f(a) + f'(a)(x-a) + 0.5 f''(a)(x-a)^2$. This quadratic polynomial provides a reasonable approximation for $f(x)$ near a. This approximation is often expressed by writing $f(x+\Delta x) \approx f(a) + f'(a)\Delta x + 0.5 f''(a)(\Delta x)^2$, where $\Delta x = x - a$.

One can define a Taylor series for a function of several variables as well. Although we will not provide the general formula for such a Taylor series here, we will point out that the second degree Taylor polynomial for a function $f(x, y)$ centered at (a, b) is given by:

$$\begin{aligned} &f(a+\Delta x, b+\Delta y) \approx \\ &\quad f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y + 0.5 f_{xx}(a,b)(\Delta x)^2 + 0.5 f_{yy}(a,b)(\Delta y)^2 + f_{xy}(a,b)\Delta x \Delta y \end{aligned}$$

Delta-Gamma-Theta Approximation

Assume the price of an option at time t is $V(S_t)$, where S_t is the time t price of the underlying stock. Let h represent a small increment in time, and let $\varepsilon = S_{t+h} - S_t$ represent the change in the price of the stock over that period of time. We can use Taylor series to develop the following approximations for $V(S_{t+h})$.

- $V(S_{t+h}) \approx V(S_t) + \Delta\varepsilon$ (Delta approximation)
- $V(S_{t+h}) \approx V(S_t) + \Delta\varepsilon + 0.5\Gamma\varepsilon^2$ (Delta-Gamma approximation)
- $V(S_{t+h}) \approx V(S_t) + \Delta\varepsilon + h\theta$ (Delta-Theta approximation)
- $V(S_{t+h}) \approx V(S_t) + \Delta\varepsilon + 0.5\Gamma\varepsilon^2 + h\theta$ (Delta-Gamma-Theta approximation(

Example 4.11

Assume the Black-Scholes framework holds. The price of a nondividend-paying stock is currently 65. You are given the following information regarding options on this stock:

- A *T*-year *K*-strike European call on the stock currently costs 13.40, has a delta of 0.5819, a gamma of 0.02084, and a theta of -4.8106.
- A *T*-year *K*-strike European put on the stock currently costs 16.27, has a delta of -0.4181, a gamma of 0.02084, and a theta of -1.4174.

One day later, the price of the stock has increased to 73. Use delta-gamma-theta approximation to estimate the new premiums for the call and the put.

Example 4.12

The current price of a non-dividend paying stock is 170. The volatility of the stock is 30% and the continuously compounded risk-free rate is 5%. A one-year 160-strike call on the stock currently costs 29.39.

One day later, the price of the stock increases to 172 and the value of the option increases to 30.76. Use delta-theta approximation to estimate theta for the call.

Example 4.13

The current price of a stock is 145. A put on this stock costs 10.34. The put has a delta of -0.4284, a gamma of 0.01353, and a theta of -2.0686.

After one week, the price of the stock has decreased. Using delta-gamma-theta approximation, you estimate that the value of the put is now 12.64. Determine the new stock price.

Example 4.14

The current price of a stock is 60. A 2-year 70-strike European call on this stock costs 2.87. The delta for the call is 0.3677 and the gamma is 0.02961. An investor purchases 1000 units of this call and delta-hedges her position.

One day later, the stock price increases to 68. Use delta-gamma-theta approximation to estimate the investor's overnight profit. Assume a continuously compounded risk-free rate of 3%.

4.4 ELASTICITY

Let S be the current price of the stock and let V be the value of an option on that stock. Assume that the stock price increases by dS, resulting in a change of dV in the value of the option. The percentage change in the value of the stock is equal to $100(dS/S)\%$ and the percentage change in the value of the option is $100(dV/V)\%$. The elasticity of the option, denoted by Ω, is the ratio of the percentage change in the value of the option as a function to the percentage change in the value of the underlying stock. In other words:

- $\Omega = \dfrac{100\,dV/V}{100\,dS/S} = \dfrac{dV/V}{dS/S} = \dfrac{dV}{dS}\dfrac{S}{V} = \dfrac{S\Delta}{V}$

This gives us the following formula for elasticity:

- $\Omega = \dfrac{S\Delta}{V}$

Several important properties of elasticity are stated below.

Properties of Elasticity

- $\Omega = \dfrac{S\Delta}{V}$
- $\Omega_{call} \geq 1$ and $\Omega_{put} \leq 0$
- The volatility of an option and the volatility of the underlying asset are related by the expression $\sigma_{option} = \sigma_{stock}|\Omega|$.
- The risk premium for a stock and an option are related by the expression $\gamma - r = \Omega(\alpha - r)$.
- The elasticity of a portfolio is the price-weighted average of the elasticity of its instruments.

Example 4.15

Assume the Black-Scholes framework applies. The current price of a stock is 210. The stock pays dividends continuously at a rate of 2% and has a volatility of 36%. The risk-free rate of interest is 6%. You are also given that $\text{Med}[S_2] = 234.51$.

Consider a of a 2-year, 240-strike European call on the stock.

a) Find the elasticity of the call.
b) Find the volatility of the call.
c) Find the continuously compounded expected yield on the call.

Consider a of a 2-year, 240-strike European put on the stock.

d) Find the elasticity of the put.
e) Find the volatility of the put.
f) Find the continuously compounded expected yield on the put.

Example 4.16

Assume the Black-Scholes framework applies. The current price of a nondividend-paying stock is 45. The volatility of the stock is less than 40%. The continuously compounded risk-free rate of interest is 5%. A 4-year, 40-strike call on this stock has a delta of 0.798562. Find the elasticity of this call.

Example 4.17

Assume the Black-Scholes framework applies. The risk-free rate of interest is 6%. The current price of a stock is 100. The stock pays dividends continuously at a rate of 2% and had a continuously compounded expected yield of 10%.

A one-year call on the stock has a premium of 16.13 and a continuously compounded expected return of 20%. Find delta for a one-year put on the stock with the same strike price as the call.

Elasticity of a Portfolio

The elasticity of a portfolio is the price-weighted average of the elasticity of its instruments. For instance, assume that a portfolio consists of n units of option A and m units of option B. Denote the option prices by A and B, and the option elasticities by Ω_A and Ω_B. The elasticity of the portfolio is given by $\Omega_\pi = \dfrac{n\,A\,\Omega_A + m\,B\,\Omega_B}{n\,A + m\,B}$.

Example 4.18

An investor creates a portfolio by purchasing 5 units of a call and 3 units of a put.

- The elasticity of the call is 2.6.
- The elasticity of the put is -3.1.
- The elasticity of the portfolio is -0.1.
- The price of the portfolio is 152.

Determine the current prices for the call and the put.

Example 4.19

The current price of a stock is 80. A one-year at-the-money European call on the stock has a premium of 10.80. A one-year at-the-money European put on the stock has a premium of 7.70.

- Portfolio A consists of 20 long calls and 15 short puts. It has an elasticity of 14.1037.
- Portfolio B consists of 20 long calls and 10 long puts. It has a delta of 8.2131.

Find the elasticity of the call and the elasticity of the put.

Example 4.20

Let S_t be the time t price of a nondividend-paying stock. Assume that $\ln[S_3/S_0]$ is normally distributed with a mean of 0.105 and a standard deviation of 0.5196.

- A one-year, 110-strike call on the stock has a premium of 21.79 and a delta of 0.7168.
- A one-year, 124-strike call on the stock has a premium of 14.69 and a delta of 0.5691.
- A one-year, 140-strike call on the stock has a premium of 8.99 and a delta of 0.4089.

A portfolio consists of 8 long 110-strike calls, 15 short 124-strike calls, and 7 long 140-strike calls. Find the volatility in the price of this portfolio, as well as the continuously compounded expected yield on the portfolio.

4.5 THE SHARPE RATIO

Sharpe Ratio

- The Sharpe ratio of an asset is defined as the ratio of its risk premium to its volatility.
- The Sharpe ratios for stocks and options are defined by $\phi_{stock} = \dfrac{\alpha - r}{\sigma_{stock}}$ and $\phi_{option} = \dfrac{\gamma - r}{\sigma_{option}}$.
- The Sharpe ratio of an option always has the same absolute value as the Sharpe ratio of the underlying asset. More specifically, $\phi_{call} = \phi_{stock}$ and $\phi_{put} = -\phi_{stock}$.

Example 4.21

Assume the Black-Scholes framework applies. A stock has a current price of 80, pays dividends continuously at a rate of 2%, and has a volatility of 30%. A two-year 75-strike put on the option has an expected return of 40%. The continuously compounded risk-free rate of interest is 5%.

a) Calculate Sharpe ratio for the put.
b) Calculate the expected return for the stock.
c) Calculate the Sharpe ratio for the stock.

Example 4.22

Assume the Black-Scholes framework applies. A nondividend-paying stock is currently worth 100. The continuously compounded risk-free rate of interest is 6%. You are given the following information about a one-year at-the-money European call on this stock:

- The probability that call will be exercised is 0.6093.
- The expected price of the stock given that the option is exercised is 1.1897 times the one-year forward price of the stock.
- Delta for the call is equal to 0.6241.

Find the Sharpe ratio for the call

Example 4.23

Prices for a nondividend-paying stock over a period of 9 months are given in the table below.

Month	Stock Price
1	50
2	55
3	58
4	53
5	56
6	52
7	48
8	45
9	53

Assuming a continuously compounded risk-free rate of interest equal to 3%, estimate the Sharpe ratio for this stock.

Example 4.24

Assume the Black-Scholes framework applies. A certain stock has a current price of 120 and a risk premium of 9%. A European put on the stock has a premium of 17.51 and a delta of -0.2937. The volatility of the option is 80.51%. Determine the Sharpe ratio for the put.

4.6 GREEKS FOR BINOMIAL TREES

We can approximate Δ, Γ, and θ for an option on a stock whose price is modeled using a binomial tree. The value of these Greeks will depend on the node at which they are calculated. We will use notation (S, t) to denote a node at time *t* at which the price is currently S.

- $\Delta(S,0) = \left[\dfrac{C_u - C_d}{Su - Sd}\right] e^{-\delta h}$
- $\Gamma(S,0) \approx \dfrac{\Delta(Su,h) - \Delta(Sd,h)}{Su - Sd}$
- $\theta(S,0)$ is found by solving the equation $C(Sud, 2h) = C(S,0) + \Delta(S,0)\epsilon + 0.5\Gamma(S,0)\epsilon^2 + 2h\theta(S,0)$, where $\epsilon = Sud - S$. This equation comes from the delta-gamma-theta approximation.

CHAPTER 5 – Exotic Options

5.1 ALL-OR-NOTHING OPTIONS

There are four types of all-or-nothing options.

- **Asset-or-Nothing Call.** Allows the holder to receive the stock without paying anything if $S_T > K$.
- **Asset-or-Nothing Put.** Allows the holder to receive the stock without paying anything if $S_T < K$.
- **Cash-or-Nothing Call.** Allows the holder to receive a fixed amount of cash if $S_T > K$.
- **Cash-or-Nothing Put.** Allows the holder to receive a fixed amount of cash if $S_T < K$.

Option	Notation	Payoff at time T	Price at time 0
Asset-or-Nothing Call (AONC)	$S \mid S > K$	S_T If $S_T > K$, 0 otherwise.	$S_0 e^{-\delta T} N(d_1)$
Asset-or-Nothing Put (AONP)	$S \mid S < K$	S_T If $S_T < K$, 0 otherwise.	$S_0 e^{-\delta T} N(-d_1)$
Cash-or-Nothing Call (CONC)	$1 \mid S > K$	1 If $S_T > K$, 0 otherwise.	$e^{-rT} N(d_2)$
Cash-or-Nothing Put (CONP)	$1 \mid S < K$	1 If $S_T < K$, 0 otherwise.	$e^{-rT} N(-d_2)$

If a cash-or-nothing option pays *c* (rather than 1) at expiration, then we will denote it by $c \mid S > K$ or $c \mid S < K$.

Relationship to Standard Options

- A *K*-strike European call is equal to $(S \mid S > K) - (K \mid S > K)$. Also note that $(S \mid S > K) = S_0 \Delta_C$.
- A *K*-strike European put is equal to $(K \mid S < K) - (S \mid S < K)$. Also note that $(S \mid S < K) = -S_0 \Delta_P$.

Example 5.1

Assume the Black-Scholes framework applies. The risk-free rate of interest is 6%. The volatility of a nondividend-paying stock is 32%. A 4-year cash-or-nothing put on this stock with a trigger price of 135 has a premium of 0.32494. Find the price of a 4-year asset-or-nothing call on this stock with a trigger price of 135.

Example 5.2

Assume the Black-Scholes framework applies. You are given the following information:

- A certain stock has a volatility of 25%.
- The volatility of a one-year 50-strike call on the stock has a volatility of 120.68%.
- A market-maker writes one unit of this call and delta hedges their position. The cost of the stock in the hedging portfolio is 34.74.

Find the price of a one-year cash-or-nothing call on the stock with a trigger price of 50.

Example 5.3

Assume the Black-Scholes framework applies.. You are giving the following information:

- A stock worth 75 pays dividends continuously at a rate of 2%.
- The stock has a volatility of 40%.
- The Sharpe ratio for the stock is 0.
- A six-month asset-or-nothing call on the stock with a trigger of 80 costs 36.65.

Calculate the probability that the asset-or-nothing call will be exercised.

5.2 GAP OPTIONS

A gap option is similar to a standard option, but involves two preset prices.

- The **Strike Price** of a gap option, K_1, is the amount paid for the stock if the option is exercised.
- The **Trigger Price** of a gap option, K_2, is the price that determines if the option will be exercised.

The payoffs and prices for gaps options are described in the table below.

Option	Payoff at time T	Price at time 0*
Gap Call	$S_T - K_1$ if $S_T > K_2$, 0 otherwise	$\text{GapCall} = S_0 e^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2)$
Gap Put	$K_1 - S_T$ if $S_T < K_2$, 0 otherwise	$\text{GapPut} = K_1 e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$

* Use K_2 when calculating d_1 and d_2 for a gap option.

Comments:

- Note that if $K_1 = K_2$, then the gap option is simply a standard option.
- Exercise on a gap option is non-elective. If the trigger condition is met, the option must be exercised.
- If $K_2 < K_1$ for a gap call, then a negative payoff will occur if $K_2 < S_T < K_1$.
- If $K_1 < K_2$ for a gap put, then a negative payoff will occur if $K_1 < S_T < K_2$.

Parity Relation: Gap options satisfy the following parity relation: $\text{GapCall} - \text{GapPut} = S_0 e^{-\delta T} - K_1 e^{-rT}$

Example 5.4

Assume the Black-Scholes framework applies. The current price of a stock is 180. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate is 6%. You are given the following information concerning options on the stock.

- A two-year, 210-strike gap call with a trigger of 200 currently costs 30.6182.
- A two-year, 190-strike gap call with a trigger of 200 currently costs 37.3276.

Find the volatility of the stock.

Example 5.5

The current price of a stock is 97. The stock pays dividends continuously at a rate of 3%. The risk-free rate of interest is 8%. A portfolio is created by purchasing 30 gap calls on the stock and selling 30 gap puts on the stock. Each of the options involved is a three-year options with a strike price of 93 and trigger price of 90. Find the value of the portfolio.

Example 5.6

Assume the Black-Scholes framework applies. You are given the following information concerning options on a stock:

- A one-year, 50-strike European put on the stock costs 10.50.
- A one-year cash-or-nothing put on the stock with a trigger of 50 costs 0.75.

Find the price of a one-year 60-strike gap put on the stock with a trigger of 50.

5.3 ASIAN OPTIONS

Arithmetic and Geometric Averages

Let $S_1,\ S_2,\ S_3,\ \ldots,\ S_n$ be prices of a stock at periodic intervals. We can average these stock prices in two ways.

- **Arithmetic Average:** $A(S) = \frac{1}{n}\sum S_i$
- **Geometric Average:** $G(S) = \sqrt[n]{\prod S_i}$

It should be noted that the inequality $G(S) \le A(S)$ always holds.

In this section, we will use $\bar{S}$ to denote an arbitrary average that could be considered as being either arithmetic or geometric.

Asian Options

An Asian option is one in which the payoff is calculated by replacing either the strike price or the stock price with $\bar{S}$, the average price of the stock over the duration of the option. The averages are calculated using stock prices observed at the end of consecutive periods of equal length.

There are three descriptors used to classify Asian options:

- The average $\bar{S}$ can be an "Average Price" option or an "Average Strike Option".
- The average $\bar{S}$ can be either a arithmetic average or a geometric average.
- The option can be either a call or a put.

As a result, there are eight different types of Asian options that can be considered. A detailed description of how Asian options work is provided in the table below.

	Average Price Asian Option	Average Strike Asian Option
Description	• $\bar{S}$ replaces S_t in the payoff formula. • The actual value of the stock at expiration is not important, only the average. • A specific strike price is still used.	• $\bar{S}$ replaces K in the payoff formula. • No strike price is specified for the option. • The option compares S_t against the average price $\bar{S}$.
Calls	The payoff of a purchased call is • $PO = \max\left[0,\ \bar{S} - K\right]$	The payoff of a purchased call is • $PO = \max\left[0,\ S_t - \bar{S}\right]$
Puts	The payoff of a purchased put is • $PO = \max\left[0,\ K - \bar{S}\right]$	The payoff of a purchased put is • $PO = \max\left[0,\ \bar{S} - S_t\right]$
Effect of n	Value of option decreases as n increases.	Value of option increases as n increases.

Example 5.7

Prices for a stock on five different dates are provided below.

Date	Jan 1	Feb 1	Mar 1	Apr 1	May 1
Price	110	100	120	80	90

Find the payoff of the following options, all of which were purchased on January 1.

a) A four-month arithmetic average price Asian call with a strike price of 100.
b) A four-month geometric average price Asian call with a strike price of 100.
c) A four-month arithmetic average strike Asian put.
d) A four-month geometric average strike Asian put.

Example 5.8

Prices for a stock at the beginning of seven consecutive months are provided below.

Date	Jan 1	Feb 1	Mar 1	Apr 1	May 1	Jun 1	Jul 1
Price	50	56	51	x	54	y	58

The payoff for a 6-month arithmetic average strike Asian call expiring on Jul 1 is 3.5.
The payoff for a 6-month, 50-strike geometric average price Asian call expiring on Jul 1 is 4.35.

Find x and y.

Pricing Asian Options with Binomial Trees

Asian options can be priced using binomial trees by calculating the average stock price along each path from the left to the right. Since the averages will depend on the path taken, the trees must not be allowed to recombine.

Example 5.9

The price of a stock currently worth 100 is modeled using a 2-period forward tree with each period being six months. The stock pays dividends continuously at a rate of 2% and has a volatility of 30%. The continuously compounded risk-free rate is 6%. Calculate the premiums for the following options on the stock.

a) A one-year geometric average strike Asian call on the stock.
b) A one-year, at-the-money geometric average price Asian call on the stock.

Arithmetic versus Geometric Averages

Consider two Asian options which use different methods of calculating the average, but are otherwise identical. Since it is known that $G(S) \le A(S)$, one can determine the relative value of these options by considering the role of the average in the formula for the payoff of the options.

5.4 BARRIER OPTIONS

Barrier options have a "barrier" or "trigger" value B specified along with the strike price K. Barrier options start out as being either "on" or "off", and can switch states once if the stock price hits the trigger at some point prior to their expiration. If the option is "on" at expiration, then the payoff is calculated as normal. There are two primary types of barrier options: Knock-In Options, and Knock-Out Options.

Knock-In Options

These options start out in the "off" state, but turn on if the stock price hits the barrier. We classify knock-in options based on the value of the barrier in relation to the current stock price.

- An *up-and-in* option is a knock-in option in which $B > S_0$.
- A *down-and-in* option is a knock-in option in which $B < S_0$.

Knock-Out Options

These options start out in the "on" state, but turn off if the stock price hits the barrier. We classify knock-out options based on the value of the barrier in relation to the current stock price.

- An *up-and-out* option is a knock-out option in which $B > S_0$.
- A *down-and-out* option is a knock-out option in which $B < S_0$.

Pricing Barrier Options with Binomial Trees

Barrier options can also be priced using binomial trees. As with Asian options, the option value at any given node is dependent on the path taken to the node, so the tree cannot be allowed to recombine.

Example 5.10

The price of a nondividend-paying stock currently worth 100 is modeled using a 3-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is four months. The continuously compounded risk-free rate is 6%. Calculate the prices for the following options on the stock.

a) A one-year up-and-in call with a strike of 110 and a barrier of 130.
b) A one-year up-and-out call with a strike of 70 and a barrier of 110.
c) A one-year down-an-in call with a strike of 110 and a barrier of 98.
d) A one-year up-and-in put with a strike of 120 and a barrier of 110.
e) A one-year down-and-in put with a strike of 100 and a barrier of 80.
f) A one-year down-and-out put with a strike of 120 and a barrier of 78.

Relationship to Ordinary Options

Barrier options are related to ordinary options in the following ways:

- The value of a barrier option is always less than or equal to the value of a similar ordinary option.
- A KI option and a KO option with the same parameters together equal a ordinary option.
- If $B \le K$, then an up-and-in call is equal to an ordinary call.
- If $B \ge K$, then a down-and-in put is equal to an ordinary put.

Example 5.11

The value of a nondividend-paying stock is currently 100. The one-year forward price of the stock is currently 108.33. The prices for several one-year at-the-money barrier options on the stock are given below.

Option	Barrier	Price
Up-and-out put	120	16.38
Up-and-out call	120	10.92
Down-and-out put	80	7.48
Down-and-in put	80	4.82
Down-and-in call	80	2.86
Down-and-out call	80	X

Find X.

Example 5.12

The current price of a nondividend-paying stock is 150. The continuously compounded risk-free rate of interest is 4%.

- The price for a two-year, 140-strike, up-and-in call with a barrier of 160 is 10.50.
- The price for a two-year, 140-strike, up-and-out call with a barrier of 160 is 14.30.

Find the price of a two-year down-and-in put with a strike of 130 and a barrier of 140.

Rebate Options

A rebate option is a special barrier option that pays a fixed amount if and only if the barrier is hit. The payment may occur at the time the barrier is hit or at expiration, depending on the terms of the option.

5.5 COMPOUND OPTIONS

A compound option is an option that has another option as its underlying asset. Since every compound option involves two options, we need to standardize some terminology and notation before continuing.

- The *compound option* is the "option on an option" and the *underlying option* is the underlying asset.
- The compound option has a strike price of x, and the underlying option has a strike of K.
- We will assume that the compound option is purchased at time 0.
- The compound option expires at time t_1 and the underlying option expires at time *T*, where $t_1 < T$.

There are four types of compound options:

- CallOnCall: Option to buy a call.
- CallOnPut: Option to buy a put.
- PutOnCall: Option to sell a call.
- PutOnPut: Option to sell a put.

Payoff of Compound Options

Let $V_u = V\left(S_{t_1},\ K,\ T - t_1\right)$ be the value of the underlying option at time t_1.

- The payoff of a compound call at time t_1 is $\max\left[0,\ V_u - x\right]$.
- The payoff of a compound put at time t_1 is $\max\left[0,\ x - V_u\right]$.

Pricing Compound Options

Compound options can be priced using binomial trees. First, use the stock underlying the underlying option to calculate the risk-neutral probabilities. Then value the underlying option at each node at time t_1 in order to find the payoff of the compound option at each node. Calculate the expected payoff, and then discount to time 0 using the risk free rate.

Example 5.13

The price of a stock currently worth 100 is modeled using a 3-period binomial tree with $u = 1.2$ and $d = 0.8$. Each period is six months. The stock pays dividends continuously at a rate of 2%. The continuously compounded risk-free rate is 5%.

Consider a 1.5 year call with a strike price of 80. Find the price of a six-month 20-strike compound put option on such a call.

Parity Relations for Compound Options

We have the following parity relationships for compound options. Assume all variables are as defined above.

- $CallOnCall - PutOnCall = Call - x e^{-r t_1}$
- $CallOnPut - PutOnPut = Put - x e^{-r t_1}$

Example 5.14

Assume the Black-Scholes framework applies. A stock currently with 80 pays dividends continuously at a rate of 2% and has a volatility of 30%. The continuously compounded risk-free rate of interest is 6%.

Consider a compound call option on a put. The underlying put expires 4 years from now and has a strike price of 90. The compound option expires in one year, has a strike price of 13, and costs 5.20.

Find the price of a compound put with the same underlying option as the compound call.

5.6 EXCHANGE OPTIONS

Exchange options provide the owner the right to exchange on asset for another upon expiration of the option. Thus, there are two assets involved in any exchange option.

- The underlying asset, which we will call *S*.
- The strike asset, which we will call *K*.

Notation for Exchange Option

We will denote the value of the assets at time 0 by S_0 and K_0. The time *T* values will be denoted S_T and K_T. We will denote the dividend rates of the two assets by δ_S and δ_K. Similarly, the volatility of the assets will be given by σ_S and σ_K. The payoff and prices for exchange options are given in the table below.

Relative Volatility

The volatility for a stock is calculated with respect to the currency in which the price of the stock is stated, and the stocks involved in an exchange option will each have their own volatility. To apply the Black-Scholes formula, we need to have a concept of how volatile the prices of the stocks are with respect to each other. Let R_S denote the annual return for the underlying asset and let R_K denote the annual return for the strike asset. We define the relative return by $R = R_S - R_K$. Let $\sigma = \sqrt{\text{Var}[R]}$. We can apply algebraic properties of the variance and covariance to obtain the expression at $\sigma^2 = \text{Var}[R] = \text{Var}[R_S + R_K] = \text{Var}[R_S] + \text{Var}[R_K] - 2\text{Cov}[R_S, R_K]$. Since $\sigma_S^2 = \text{Var}[R_S]$, and $\sigma_K^2 = \text{Var}[R_K]$, we can write this expression as $\sigma^2 = \sigma_S + \sigma_K - \text{Cov}[R_S, R_K]$. Now, let ρ denote the correlation coefficient of R_S and R_K. Then $\text{Cov}[R_S, R_K] = \rho\sigma_S\sigma_K$ and $\sigma^2 = \sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K$.

Pricing Exchange Options

Consider stocks *S* and *K*. Let S_T and K_T denote the time *T* prices of the two stocks. Assume that stock *S* has a volatility of σ_S and pays dividends at at rate of δ_S. Assume that stock *K* has a volatility of σ_K and pays dividends at a rate of δ_K. Let the correlation coefficient for the two stocks be given by ρ. Consider an exchange option with *n* shares of stock *S* as the underlying asset and *m* shares of stock *K* as the strike asset. We can use the Black-Scholes formula to price such an option as follows:

- Let $\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K}$.
- Let $d_1 = \dfrac{\ln(nS_0 / mK_0) + (\delta_K - \delta_S + 0.5\sigma^2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.
- The price of the exchange call is given by $C = nS_0e^{-\delta_S T}N(d_1) - mK_0e^{-\delta_K T}N(d_2)$.
- The price of the exchange put is given by $P = mK_0e^{-\delta_K T}N(-d_2) - nS_0e^{-\delta_S T}N(-d_1)$.

Example 5.15

Assume the Black-Scholes framework applies. You are given the following information about stocks *A* and *B*.

- Stock *A* has a price of 110 and a volatility of 30%. It pays dividends at a rate of 4%.
- Stock *B* has a price of 100 and a volatility of 20%. It does not pay dividends.
- The correlation between the two stocks is 0.65.

Find the price of an option that gives the owner the right to exchange one share of Stock *B* for one share of Stock *A* two years from now.

Example 5.16

Assume the Black-Scholes framework applies. You are given the following information about stocks A and B.

- Stock A has a price of 120 and a volatility of 20%. It pays dividends at a rate of 2%.
- Stock B has a price of X and a volatility of 30%. It pays dividends at a rate of 2%.
- The correlation between the two stocks is -0.42.

Find the price of an option that gives the owner the right to exchange $120/X$ shares of Stock B for one share of Stock A one year from now.

Example 5.17

Assume the Black-Scholes framework applies. You are given the following information:

- The Euro-dollar exchange rate is \$1.50 per Euro. The volatility of this rate is 12%.
- The pound-dollar exchange rate is \$1.80 per pound. The volatility of this rate is 18%.
- The risk-free rate of interest for both Euros and pounds is 2%.
- The correlation coefficient between the returns for Euros and pounds is 0.6.

Find the price, in pounds, of an option that allows the owner to purchase 100 Euro for 74 pounds two years from now.

Parity Relation

Exchange options satisfy the following parity relation: $\text{Call} - \text{Put} = S_0 e^{-\delta_S T} - K_0 e^{-\delta_K T}$

Option Duality

Let A and B each represent assets. Consider an exchange option that allows the holder of the option to give up Asset A in exchange for Asset B at expiration. This option could be viewed in two different ways:

- It can be seen as a call in which Asset A represents the strike price and Asset B represents the underlying asset. Denote the premium of this call by $C(S=A,\ K=B)$.
- It can be seen as a put in which Asset A represents the underlying asset and Asset B represents the strike price. Denote the premium of this put by $P(S=A,\ K=B)$.

The values $C(S=A,\ K=B)$ and $P(S=A,\ K=B)$ each denote the premium of the same option. As a result, we see that $\text{Call}(S=A,\ K=B) = \text{Put}(S=B,\ K=A)$.

5.7 MAXIMA AND MINIMA

A standard call has a payoff of the form $\max[0, S_T - K]$ and a standard put has a payoff of $\max[0, K - S_T]$. If an exotic option has a payoff of the form $\max[g(S_T), f(S_T)]$ or $\min[g(S_T), f(S_T)]$, where g and f are linear functions, then the price of the option can be expressed in terms of a put or a call by using the following rules:

- $\max[A, B] = B + \max[A - B, 0] = A + \max[0, B - A]$
- $\min[A, B] = B + \min[A - B, 0] = A + \min[0, B - A]$
- $\max[kA, kB] = k\max[A, B]$ if $k > 0$
- $\min[kA, kB] = k\min[A, B]$ if $k > 0$
- $\max[-A, -B] = -\min[A, B]$
- $\max[A, B] + \min[A, B] = A + B$, and so $\min[A, B] = A + B - \max[A, B]$

Example 5.18

Assume the Black-Scholes framework applies. The current price of a nondividend-paying stock is 25. The stock has a volatility of 40%. The continuously compounded risk-free rate of interest is 4%. Consider an option that pays the owner the smaller of the following two values at the end of two years: 200, or 10 times the price of the stock at that time. Calculate the price of this option.

Example 5.19

Assume the Black-Scholes framework applies. You are given the following information regarding stocks X and Y.

- Both stocks are currently worth 75.
- Both stocks pay dividends continuously at a rate of 3%.
- Stock X has a volatility of 35%. Stock Y has a volatility of 25%.
- The correlation between the returns for the two stocks is 20%.

A financial derivative allows the purchaser to pay a premium today in exchange for their choice of one of the two stocks, to be delivered after two years. The selection of the stock will be made at the time of delivery. Calculate the premium for this option.

5.8 CHOOSER OPTIONS

A chooser option (also known as an as-you-like-it option) allows the owner to decide at time t whether he would like for the option to be a K-strike call, or a K-strike put, either of which will expire at time T. The underlying stock, strike price, exercise date, and premium are all determined when the option is purchased. The time t at which the holder of the option has to decide between a call or a put is also decided when the contract is initiated.

Pricing Chooser Options

We will now develop a formula for pricing chooser options. Notice that the value of the option at time t will be

$$V_t = \max\left[\text{Call}\left(S_t,\ K,\ T-t\right),\ \text{Put}\left(S_t,\ K,\ T-t\right)\right]$$

We can use rules from the previous section to rewrite this expression as

$$V_t = \text{Call}\left(S_t,\ K,\ T-t\right) + \max\left[0,\ \text{Put}\left(S_t,\ K,\ T-t\right) - \text{Call}\left(S_t,\ K,\ T-t\right)\right]$$

We now apply put-call parity to obtain

$$V_t = \text{Call}\left(S_t,\ K,\ T-t\right) + \max\left[0,\ K e^{-r(T-t)} - S_t e^{-\delta(T-t)}\right]$$

Again using rules from Section 5.7, we obtain

$$V_t = \text{Call}\left(S_t,\ K,\ T-t\right) + e^{-\delta(T-t)}\max\left[0,\ K e^{-(r-\delta)(T-t)} - S_t\right]$$

Notice that $\max\left[0,\ K e^{-(r-\delta)(T-t)} - S_t\right]$ is the payoff of a t-year put with a strike price of $K e^{-(r-\delta)(T-t)}$. Also note that the $\text{Call}\left(S_t,\ K,\ T-t\right)$ refers to the premium of a European call purchased at time t and expiring at time T. It follows from these observations that purchasing a chooser option is equivalent to purchasing a T-year, K-strike call, along with $e^{-\delta(T-t)}$ units of a t-year put with a strike price of $K e^{-(r-\delta)(T-t)}$. This is summarized below.

Pricing Chooser Options

Consider a T-year, K-strike chooser option. Assume the owner choices if the option will be a call or a put at time t. Then the price of the option at time 0 is given by:

- $V = \text{Call}\left(S_0,\ K,\ T\right) + e^{-\delta(T-t)}\text{Put}\left(S_0,\ K e^{-(r-\delta)(T-t)},\ t\right)$

Example 5.20

Assume the Black-Scholes framework applies. The current price of a stock is 100. The stock pays dividends continuously at a rate of 2% and has a volatility of 30%. The continuously compounded risk-free rate of interest is 6%. Find the price of a chooser option that expires in one year, and with a decision date 6 months from now.

Example 5.21

Assume the Black-Scholes framework applies. The current price of a nondividend-paying stock is 110. You are given the following information about derivatives on this stock:

- The two-year forward price on the stock is 121.88. The price of a two-year
- A two-year, 100-strike European put on the stock currently costs 9.70.
- A two-year, 100-strike chooser option on the stock with a decision date occurring in one year currently costs 34.63.

Find the price of a one-year, 95-strike European put on the stock.

5.9 FORWARD START OPTIONS

In a forward start option, the strike price is not when the option is entered into, but rather at some predetermined point in time between when the option is purchased and when it expires. At that time, the strike price is set to be equal to the current stock price times some multiplier that is determined at the time of purchase.

Pricing Forward Start Options

Consider a forward start option purchased at time 0 and expiring at time *T*. Assume that at some time $t \in (0, T)$ the strike price will be set at $K = c\,S_t$ for some known multiplier *c*.

We can apply the Black-Scholes formula, to determine the time *t* value of a forward strike call or put as follows.

- Let $d_1 = \dfrac{-\ln c + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$ and $d_2 = d_1 - \sigma\sqrt{T - t}$.
- The time *t* value of the forward start call is $C_{FS,t} = S_t e^{-\delta(T-t)} N(d_1) - c\,S_t e^{-r(T-t)} N(d_2)$.
- The time *t* value of the forward start put is $P_{FS,t} = c\,S_t e^{-r(T-t)} N(-d_2) - S_t e^{-\delta(T-t)} N(-d_1)$.

We can pull the time *t* values of these options back to time 0 to calculate the premiums. We do this by replacing S_t with the prepaid forward price $S_{0,t}^P = S_0 e^{-\delta t}$ and discounting the values to time 0 by multiplying by e^{-rt}. This yields the following expressions for the premiums for forward strike options.

Pricing Forward Start Options

Consider a forward start option purchased at time 0 and expiring at time *T*. Assume that at some time $t \in (0, T)$ the strike price will be set at $K = c\,S_t$ for some known multiplier *c*. The prices of such an options can be calculated as follows.

- Let $d_1 = \dfrac{-\ln c + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$ and $d_2 = d_1 - \sigma\sqrt{T - t}$.
- The price of a forward start call at time 0 is $C_{FS} = S_0 e^{-\delta T} N(d_1) - c\,S_0 e^{-r(T-t) - \delta t} N(d_2)$.
- The price of a forward start put at time 0 is $P_{FS} = c\,S_0 e^{-r(T-t) - \delta t} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$.

Example 5.22

Assume the Black-Scholes framework applies. A stock pays dividends continuously at a rate of 2% and has a volatility of 30%. The current price of the stock is 160. The continuously compounded risk-free rate of interest is 4%. You purchase a forward start call on this stock. The call expires in three years. After one year, the strike price of the call will be set to be equal to 110% of the price of the stock at that time. Price this option.

Example 5.23

Assume the Black-Scholes framework applies. The price of one share of a stock is currently 80. The Sharpe ratio for the stock is 0. The continuously compounded risk-free rate of interest is 3%. consider a forward start option on the stock that provides the owner with a one-year at-the-money put six months from now. The price of this option is 8.5153. Determine the volatility of the stock.

CHAPTER 6 – Monte Carlo Valuation

6.1 SIMULATING STOCK PRICES

Monte Carlo Simulation

When there is not a convenient formula available for pricing a particular derivative instrument, we can resort to using computer simulation to value the derivative. The process works like this:

1. Select a risk-neutral statistical model for the stock price. We will use the risk-neutral lognormal model.
2. Generate several simulated observations of the future stock price according to the model.
3. For each observation, calculate the payoff of the derivative.
4. Average the payoffs across all simulated observations.
5. Discount the average payoff using the risk free rate to obtain an estimate for the option price.

Generating Standard Normal Random Numbers

To generate simulated stock prices according to the lognormal model, we must first be able to generate random values according to a normal distribution. We will discuss two methods of doing so, both of which rely on first generating observations of a uniformly distributed random variable $U \sim Uniform(0,1)$. Most programming languages and software packages include methods for generating random values according to a uniform distribution.

The two methods we use for generating random observations of $Z \sim Normal(0,\ 1)$ are as follows:

1. Let $u_1,\ u_2,\ \ldots,u_{12}$ be 12 randomly generated observations of $U \sim Uniform(0,1)$. Now, let $z = \sum_{i=1}^{12} u_j - 6$. It can be shown that values of z generated in this way will have a mean of 0 and a variance of 1. By the Central Limit Theorem, they will also be approximately normal. Thus, any value z generated in this way can be considered to be an observation of $Z \sim Normal(0,\ 1)$.
2. Let u be a single observation of $U \sim Uniform(0,1)$. Let $z = N^{-1}(u)$. Then z will be an observation of $Z \sim Normal(0,\ 1)$.

Generating Lognormal Random Numbers

We will now discuss how to generate random values following a lognormal distribution with parameters *m* and *v*. Let $N \sim Normal(m,\ v^2)$ and $X = e^N \sim LogN(m,\ v^2)$. Notice that $N = m + v\,Z$, where $Z \sim Normal(0,\ 1)$.

- Let z be an observation of Z.
- Then $n = m + v\,z$ is an observation of the normally distributed random variable N.
- Thus $e^n = e^{m+vz}$ is an observation of the lognormally distributed variable X.

Risk-Neutral Model vs. True Model

The model we use when simulating stock prices depends on our intended goal.

- If we are interested in the true expected value of the stock, or the true expected payoff of an option, then we will use the true distribution, setting $m = (\alpha - \delta - 0.5\sigma^2)T$ and $v = \sigma\sqrt{T}$ in our lognormal model.
- If we are interested in pricing an option on the stock, then discounting is required. In this case, we make an assumption of risk-neutrality, setting $m = (r - \delta - 0.5\,\sigma^2)T$ and $v = \sigma\sqrt{T}$.

Simulating Stock Prices

Assume a stock has a current price of S_0, and that we wish to generate *N* simulated prices for the stock at time *T*. Then we will generate *N* observations of $Z \sim Normal(0, 1)$, denoted by $z_1, z_2, \ldots, z_n$. For each observation of Z, we will set $S_{T,i} = S_0 e^{m + z_i v}$. The value of *m* used depends whether we are using the true or risk-neutral model, as explained above.

Example 6.1

Assume the Black-Scholes framework applies. The current price of a stock is 50. The stock pays dividends continuously at a rate of 2% and has a continuously compounded expected yield of 10%. The volatility of the stock is 30%.

Four observations of a random variable that is uniformly distributed on the interval $(0, 1)$ are generated. The observations are 0.2420, 0.6554, 0.4207, and 0.8849.

a) Use these observations to simulate four values of the price of the stock after two years.

b) Use the four simulated stock prices to estimate the expected payoff for a two-year at-the-money European call on the stock.

Simulating a Sequence of Stock Prices

We will occasionally want to simulate a sequence of stock prices at specific intervals over a period of time, as opposed to simulating a single stock price at some specific point in time. for example, this will be necessary when using simulation to price path dependent options, such as Asian or barrier options. Such a sequence of simulated stock prices is called a **run**.

To generate a run over the interval $[0, T]$, we first split the interval into *k* subintervals, each with length *h*. Let $z_1, z_2, \ldots, z_k$ be randomly generated observations of $Z \sim Normal(0, 1)$. We create our run iteratively as follows:

$$S_h = S_0 e^{m + z_1 v}, \quad S_{2h} = S_h e^{m + z_2 v}, \quad S_{3h} = S_{2h} e^{m + z_3 v}, \ldots, \quad S_T = S_{(k-1)h} e^{m + z_k v}$$

Each run will generate one stock price at time *T* as well as a sequence of stock prices at various times along the way. If we wish to generate *N* simulated prices, it will require kN standard normal random numbers.

Example 6.2

Assume the Black-Scholes framework applies. The current price of a stock is 50. The stock pays dividends continuously at a rate of 2% and has a continuously compounded expected yield of 10%. The volatility of the stock is 30%.

Four observations of a random variable that is uniformly distributed on the interval $(0, 1)$ are generated. The observations are 0.2420, 0.6554, 0.4207, and 0.8849.

a) Use these observations to simulate a run of quarterly prices for the stock over a period of one year.

b) Calculate the payoff for a one-year arithmetic average strike Asian call for the run generated in Part (a).

c) Calculate the payoff for a one-year, at-the-money arithmetic average price Asian call for the run generated in Part (a).

6.2 MONTE CARLO VALUATION FOR OPTIONS

When we lack a convenient formula for pricing a certain type of option, we can use simulated stock prices to estimate a price for the option. The process is detailed below.

1. Generate N simulated values of the stock price at time T using a risk-neutral model. If the option in question is path dependent, then generate N runs rather than N individual prices.
2. Calculate the payoff of the option for each simulated stock price, or for each run.
3. Calculate the mean payoff of the option. Denote this by $\bar{V}$.
4. Discount the mean payoff to time 0 using the risk-free rate to obtain a simulated price of $V = \bar{V} e^{-rT}$.

This process of using simulation to estimate option premiums is called **Monte Carlo valuation**.

Example 6.3

Assume the Black-Scholes framework applies. The current price of a stock is 80. The stock pays dividends continuously at a rate of 2% and has a continuously compounded expected yield of 11%. The volatility of the stock is 35%. The continuously compounded risk-free rate of interest is 4%.

Four observations of a random variable that is uniformly distributed on the interval $(0, 1)$ are generated. The observations are 0.8152, 0.3420, 0.1626, and 0.9713.

You these observations to along with Monte Carlo simulation to estimate the price of a six-month, 90 strike European call.

Example 6.4

Assume the Black-Scholes framework applies. The current price of a nondividend-paying stock is 120. The stock has a continuously compounded expected yield of 12%. The volatility of the stock is 30%. The continuously compounded risk-free rate of interest is 5%.

An option on this stock that pays $\left[(S_1 - 120)^{1.2},\ 0\right]$ at the end of one year.

Use Monte Carlo simulation to estimate the price of this option. Use the following four draws from a random variable that is uniformly distributed on the interval $(0, 1)$:

$$0.3415,\ 0.7577,\ 0.3671,\ 0.8528$$

6.3 CONTROL VARIATE METHOD

Basic Control Variate Method

If we would like to reduce the variance in the sampling distribution for our simulated values (thus reducing the margin of error in our estimate) we can use the control variate method.

- Let V be an option whose value we wish to approximate using a Monte Carlo simulation.
- Let K be an option whose value is highly correlated with that of V, and one for which we have a pricing formula. So, we know the true price of K.
- While conducting the simulation to estimate the price of V, also estimate the price of K using the same simulated stock prices. Let $\bar{V}$ and $\bar{K}$ be the price estimates generated by the simulation.
- The control variate estimate of the value of V is given by $V^* = \bar{V} + K - \bar{K}$.

Example 6.5

The current price of a nondividend-paying stock is 50. The continuously compounded risk free rate is 6%. The price of a 4-month geometric average strike Asian call on the stock is currently 4.50.

You decide to estimate the price of a 4-month arithmetic average strike Asian call on the stock using Monte Carlo simulation. You also decide to apply the basic control variate method, with the geometric average strike option being the control variable.

The results of three simulated runs for the stock are shown below.

Run	$S_{0.25}$	$S_{0.50}$	$S_{0.75}$	S_1
1	45.13	48.26	42.34	56.82
2	48.89	53.46	55.55	60.13
3	52.87	56.68	57.29	45.38

Find the simulated price for the arithmetic average strike call.

Notice that the variance of the estimate V^* is given by $\mathrm{Var}\left[V^*\right] = \mathrm{Var}\left[\bar{V}\right] + \mathrm{Var}\left[\bar{K}\right] - 2\mathrm{Cov}\left[\bar{V},\ \bar{K}\right]$. If the values of $\bar{V}$ and $\bar{K}$ are highly correlated, then it will be true that $\mathrm{Var}\left[V^*\right] < \mathrm{Var}\left[\bar{V}\right]$, resulting in the a reduction in the variance of the estimate.

Example 6.6

Consider two similar options: Option A and Option B. Assume that there exists a pricing formula for Option B, but no such formula for Option A. Monte Carlo simulation is used to estimate the price of an Option A. The basic control variate method is used, with Option B as the control variable. You are given:

- The naive Monte Carlo estimate for the price of Option A has a variance of 7.6.
- The naive Monte Carlo estimate for the price of Option B has a variance of 5.2.
- The correlation coefficient of the two estimated prices is 0.8.

Calculate the reduction in variance obtain by using the control variate method.

Boyle Modification

Generalizing on the idea basic control variate method, assume we estimate the value of V using an expression of the form $V' = \bar{V} + \beta\left(K - \bar{K}\right)$. Notice that the variance in our modified estimate V' is given by:

$$\text{Var}\left[V'\right] = \text{Var}\left[\bar{V}\right] + \beta^2\,\text{Var}\left[\bar{K}\right] - 2\beta\,\text{Cov}\left[\bar{V}\,,\,\bar{K}\right]$$

And the overall variance reduction is equal to:

$$\text{Var}\left[\bar{V}\right] - \text{Var}\left[V'\right] = -\beta^2\,\text{Var}\left[\bar{K}\right] + 2\beta\,\text{Cov}\left[\bar{V}\,,\,\bar{K}\right]$$

The reduction in variance is a quadratic function of β, and attains its maximum at $\beta = \dfrac{\text{Cov}\left[\bar{V}\,,\,\bar{K}\right]}{\text{Var}\left[\bar{K}\right]}$.

The **Boyle modification** V^{BM} is the estimate of the form $V^{BM} = \bar{V} + \beta\left(K - \bar{K}\right)$, where β is chosen to maximize the reduction in variance.

Boyle Modification

- Let V be an option whose value we wish to approximate using a Monte Carlo simulation.
- Let K be an option whose value is highly correlated with that of V, and one for which we have a pricing formula.
- Let $\bar{V}$ and $\bar{K}$ denote the naive estimates for V and K obtained by the simulation.
- The Boyle modification estimate for V is given by $V^{BM} = \bar{V} + \beta\left(K - \bar{K}\right)$, with β given by:
  $$\beta = \frac{\text{Cov}\left[V\,,\,K\right]}{\text{Var}\left[K\right]} = \frac{E\left[V\,K\right] - E\left[V\right]E\left[K\right]}{E\left[K^2\right] - E\left[K\right]^2} = \frac{\left(\sum v_i\,k_i\right) - n\,\bar{V}\,\bar{K}}{\left(\sum k_i^2\right) - n\,\bar{K}^2}$$
- Note that $\text{Var}\left[V^{BM}\right] = \text{Var}\left[\bar{V}\right] + \beta^2\,\text{Var}\left[\bar{K}\right] - 2\beta\,\text{Cov}\left[\bar{V}\,,\,\bar{K}\right] = \text{Var}\left[\bar{V}\right]\left(1 - \rho_{\bar{V}\bar{K}}^2\right)$.

Example 6.7

Consider two similar options: Option X and Option Y. Assume that there exists a pricing formula for Option X, but no such formula for Option Y. Monte Carlo simulation is used to estimate the price of an Option Y. A control variate method is used, with Option X as the control variable. Let $\bar{X}$ and $\bar{Y}$ denote the naive estimates resulting from the simulation. You are given that $\text{Var}\left[\bar{X}\right] = 1.44$, $\text{Var}\left[\bar{Y}\right] = 2.56$, and $\rho_{\bar{X}\bar{Y}} = 0.63$.

Assume that the final estimate for Y will have the form $Y' = \bar{Y} + \beta\left(X - \bar{X}\right)$.

a) Find the value of β which will minimize $\text{Var}\left[Y'\right]$.

b) Determine the reduction in variance obtained by using this estimate rather than the naive estimate.

Example 6.8

Let A denote the price of an arithmetic average strike Asian put on a stock. Let G be the price of an otherwise equivalent geometric average strike Asian put on the same stock. A pricing formula exists for finding G, which has a premium of 18.27. No pricing formula exists for A.

You plan on using Monte Carlo simulation to estimate the value of A, using the geometric average option as a control variable. Assume the Boyle method is used. The results of the simulation are as follows:

- The naive estimates for A and G are $\bar{A} = 15.60$ and $\bar{G} = 20.42$.
- The variances for the estimates were $\text{Var}[\bar{A}] = 1.82$ and $\text{Var}[\bar{G}] = 1.40$.
- The final estimate obtained for A is $A^{BM} = 14.31$.

Find $\text{Var}\left[A^{BM}\right]$.

Using a Calculator to Find β

Note: If you are given the observations (k_i, v_i), then you can easily find β using the TI-30XS. Enter the k and v values in **L1** and **L2**, respectively. Then press **[stat]**, **[2-Var Stats]**, and select **L1** and **L2**. Then β will be the value that shows up for a for the regression line. That said, you still need know the various formulas for β in case you are asked to calculate β given information other than the actual observations.

Example 6.9

Consider two options on the same stock:

- Option A is a standard one-year, 100-strike European put on a stock.
- Option B is a special one-year, 100-strike European put whose payoff is equal to the payoff of Option A raised to the power 1.1.

The Black-Scholes formula is applied to determine that the price of the standard put is 4.60. Monte Carlo simulation is used to simulated the price of Option B. The estimate obtained has for B has the form $B^* = \bar{B} + \beta\left(A - \bar{A}\right)$, where β is chosen to minimize the variance of the estimate.

Six simulated stock prices are generated: 156.34, 87.46, 95.02, 125.68, 139.56, and 91.65.

Calculate B^* and $\text{Var}\left[B^*\right]$. Assume a continuously compounded risk free rate of 4%.

6.4 OTHER VARIANCE REDUCTION TECHNIQUES

We now discuss two methods of increasing our sample size without obtaining additional random observations.

- Assume $u_1, u_2, \ldots, u_n$ have been generated.

Antithetic Variates

- For each *i*, add $u_i^* = 1 - u_i$ to the sample.
- For each *i*, add $n_i^* = -n_i$ to the sample.

Stratified Sampling

- For each *i*, add $u_i^* = \frac{i-1}{n} + \frac{u_i}{n}$ to the sample.
- In effect, stratified sampling breaks the interval $[0, 1]$ into *n* subintervals, and then uses u_i to generate a value from the *i*th subinterval.

Example 6.10

Eight random draws of a variable that is uniformly distributed on the interval $(0, 1)$ are generated. These draws are:

0.70	0.93	0.48	0.35	0.47	0.89	0.08	0.97

Use stratified sampling with four subintervals to generate eight more draws.

CHAPTER 7 – Binomial Tree Models for Interest Rates

7.1 GENERAL TREE MODELS

In this lesson, we will use binomial trees to model changing short-term interest rates. We will also use these interest rate models to price bonds. We begin with a discussion of notation.

Bond Price Notation

We will use the following notation when referring to the prices of zero-coupon bonds.

- Let $P(t, T)$ denote the price determined at time *t*, and paid at time *t*, of a bond maturing for $1 at time *T*.
- Let $P_0(t, T)$ denote the price determined at time 0, but paid at time *t*, of a bond maturing for $1 at time *T*.

We can make the following observations about these bond prices.

- Notice that $P_0(t, T)$ is the forward price of $P(t, T)$. As such, we also denote $P_0(t, T)$ by $F_{0,T}(t, T)$.
- Interest theory concepts tell us that $P_0(t, T) = \dfrac{P(0, T)}{P(0, t)}$.
- If risk free rate is constant, then $P(0, T) = e^{-rT}$. Even if the risk free rate varies, we can use $P(0, T)$ as a present value factor. The present value of a payment of K occurring at time *T* would be $P(0, T) \cdot K$.
- These bond prices are related to spot rates as follows: $P(0, T) = \dfrac{1}{(1 + s_T)^T}$.

Structure of a Binomial Tree Interest Rate Model

Binomial trees can be used to model changes in short term interest rates over time. The details of how tree rate models work are provided in the following comments.

- Each node in the tree will represent the interest rate during a period of length *h*. Typically, *h* will be 1 year. It is important to remember that in interest rate models nodes represent an entire period, not a particular moment in time.
- The process used for pricing bonds using tree rate models will be path-dependent. As such, our binomial trees must not be allowed to recombine.
- Some method will be provided to determine the magnitude of an up-move or down-move in the rates when moving from one period to the next. Risk-neutral probabilities of an up-move or a down-move will also be provided.
- Unless otherwise specified, rates will be continuously compounded. There are exceptions to this rule, however. For example, the Black-Derman-Toy model that we will study in the next section makes use of annual effective rates rather than continuously compounded rates.

Pricing Bonds Using the Binomial Tree Model

We use the following process for pricing a zero-coupon bond using a binomial tree model.

1. For each path through the binomial tree, we use the rates along that path to discount the face value of the bond back to time 0, thus obtaining one hypothetical price for each path.
2. We then calculated a weighted average of our hypothetical bond prices, with the weights determined by the probability of each path occurring. This weighted average will be our bond price.

Take note of the following comments regarding this process.

- Since each node represents an entire period rather than a single moment in time, an n period tree rate model will only fork n-1 times. For example, a three-year binomial tree rate model with one-year periods will only fork twice.
- If the rates in our model are continuously compounded, then we can add the rates at each node along a given path to get the total rate to discount by. This is not true when working with annual effective rates.
- It is not valid to calculate the "expected yield rate" for each period and then use those rates to discount the face value back to time 0. That will result in a different (incorrect) bond price.

Example 7.1

A three-period binomial tree interest rate model is constructed with each period being one year. The initial interest rate is 5%. The rate will either increase or decreased by 1.5% each period, with the risk-neutral probability of an increase being equal to 55%.

a) Determine the price of a three-year, 1000-par zero-coupon bond using this model.

b) Determine the price of a two-year, 1000-par zero-coupon bond using this model.

c) Determine the one-year forward price for a two-year 1000-par zero-coupon bond.

We can use binomial tree rate models to price options on bonds, as illustrated in the following example.

Example 7.2

A three-period binomial tree interest rate model is constructed with each period being one year. The initial interest rate is 6%. The rate will either increase or decreased by 1% each period, with the risk-neutral probability of an increase being equal to 60%.

a) Determine the price of a two-year, 950-strike European call on a one-year, 1000-par zero-coupon bond.

b) Determine the price of a two-year, 950-strike American call on a one-year, 1000-par zero-coupon bond.

7.2 THE BLACK-DERMAN-TOY MODEL

Characteristics of the Black-Derman-Toy Model

The Black-Derman-Toy model is a specific binomial rate tree model with the following characteristics.

- Each period has length *h* (usually 1 year).
- The rates provided are annual effective rather than continuously compounded.
- The risk-neutral probability of an up-move is $p^* = 0.5$.
- Let $r_{t,i}$ be the rate for period *t* node that is positioned *i* nodes above the bottom node.
- The rates along the bottom path $\left(r_{1,0},\ r_{2,0},\ r_{3,0},\ \ldots\right)$ are generally provided.
- Each period will have its own short term volatility, which will be denoted by σ_t .
- Given the rate at a node $r_{t,i}$, the rate at the next node above is obtained by the formula $r_{t,i+1} = r_{t,i}e^{2\sigma_t\sqrt{h}}$.
- There are no formulas relating the rates from two different periods. However, if we know two rates from the same period, we can use the short term volatility to determine all of the other rates for that period.

Example 7.3

Use the incomplete Black-Derman-Toy interest rate model provided below to find the price of a 1000-par, three-year zero-coupon bond. Assume that each period represents one year.

		$r_{2,2} = 0.06$
	$r_{1,1} = 0.05$	
$r_{0,0} = 0.04$		$r_{2,1} = x$
	$r_{1,0} = 0.03$	
		$r_{2,0} = 0.02$

Example 7.4

You are given the following information regarding a three-period Black-Derman-Toy model.

- Each period is one year.
- $r_{0,0} = 0.05$, $r_{1,0} = 0.04$, and $r_{2,0} = 0.03$.
- $\sigma_1 = 0.20$ and $\sigma_2 = 0.25$.

Find the two-year forward price for a 1000-par, one-year zero-coupon bond.

Example 7.5

Assume that short-term rates are modeled using a Black-Derman-Toy tree model, with each period being one year. The current interest rate is 6%. The short term volatility in bond prices during the first year is 20%. The price of a two-year, 1000-par zero-coupon bond is currently 892.16. Find $r_{1,0}$ and $r_{1,1}$.

Long-Term Volatility

- Assume a bond maturing for 1 at time T is purchased at time 0 for a price of $P(0,T)$.
- The value of the bond after one year will be $P(1,T)$
- $P(1,T)$ has two possible values, which we will denote by $P_u(1,T)$ and $P_d(1,T)$.
- Let $y_u(1,T)$ be the annual effective yield rate obtained by purchasing the bond at time 1 for $P_u(1,T)$. Similarly, let $y_d(1,T)$ be the annual effective yield rate obtained by purchasing the bond for $P_d(1,T)$.
- Then $y_u(1,T) = \dfrac{1}{P_u(1,T)^{T-1}} - 1$ and $y_d(1,T) = \dfrac{1}{P_d(1,T)^{T-1}} - 1$.
- The volatility in the price of $P(1,T)$ is given by $\sigma_{1,T} = \dfrac{1}{(T-1)\sqrt{h}} \ln\left[\dfrac{y_u(1,T)}{y_d(1,T)}\right]$.

Example 7.6

Consider the following Black-Derman-Toy interest rate tree model.

$$
\begin{array}{ccc}
 & & r_{2,2} = 0.1024 \\
 & r_{1,1} = 0.08 & \\
r_{0,0} = 0.06 & & r_{2,1} = x \\
 & r_{1,0} = 0.05 & \\
 & & r_{2,0} = 0.04
\end{array}
$$

Find the volatility in year-one for a three-year zero-coupon bond.

7.3 PRICING CAPS

Assume that an individual borrows money at a floating rate and makes interest rate payments at the end of each period. An interest rate cap is a contract that guarantees that the borrower will not have to pay more than a certain fixed rate, regardless of what the short-term rate actually is during that period. The borrower must pay a premium to obtain a cap. We can use binomial rate trees to price caps.

Throughout this section, we will use L to denote the loan amount. This is also called the **notional value**. We will also use r_k to denote the fixed interest rate cap.

Caplets

Given any node, we can calculate the value of the cap to the borrower during the period represented by that node. These values are called caplets. The first step in pricing a cap is to determine the value of the caplets at each of the nodes in the tree. The value of the caplet for the node $r_{t,i}$ will be denoted by $C_{t,i}$ and is calculated as follows:

- If $r_{t,i} \le r_k$, the the value of the caplet at this node is $C_{t,i} = 0$.
- If $r_{t,i} > r_k$, the the value of the caplet at this node is $C_{t,i} = L(r_{t,i} - r_k)$.
- In general, the value of the caplet is $C_{t,i} = \max\left[0,\ L\left(r_{t,i} - r_k\right)\right]$.
- Note that interest payments are made at the end of the period. Thus, the value found above is the value of the caplet at *the end* of the period represented by the node in question.

Pricing Caps

The value of a cap is the probability weighted sum of the present value of all caplets. The following comments are important to keep in mind when pricing caplets.

- It is necessary to discount the value of ALL of the caplets. It is not enough to work only with caplets at the terminal nodes.
- If your rate tree is drawn in so as to allow recombination of nodes, then when discounting a caplet to find its present value, you must discount along each path leading to that node.
- Since the interest payments occur at the end of the period, it is necessary to discount each caplet through its own node when finding its present value.

Example 7.7

Consider the following Black-Derman-Toy interest rate tree model.

$$r_{2,2} = 0.16$$

$$r_{1,1} = 0.11$$

$$r_{0,0} = 0.07 \qquad\qquad r_{2,1} = x$$

$$r_{1,0} = 0.06$$

$$r_{2,0} = 0.04$$

Find the price of a 3-year interest rate cap with a cap rate of 7.5% and a notional value of 2000.

Example 7.8

Consider the following Black-Derman-Toy interest rate tree model.

			$r_{3,3} = 0.2048$
		$r_{2,2} = x$	
	$r_{1,1} = 0.098$		$r_{3,2} = y$
$r_{0,0} = 0.08$		$r_{2,1} = 0.102$	
	$r_{1,0} = 0.072$		$r_{3,1} = z$
		$r_{2,0} = 0.06$	
			$r_{3,0} = 0.05$

Consider a four-year interest rate cap with a cap rate of 9% and a notional value of 1000.

a) Find the value of the year four caplet.

b) Find the price of the cap.