HW 4.2(b) Key

1. Let *X* be a continuous random variable with probability density function given by $f(x) = K\,x^{2.2}$ for $0 \le x \le 1$, and 0 otherwise. Find the 35th-percentile of *X*, $\pi_{0.35}$.

$\boxed{\text{A)}}$ 0.7203 B) 0.6987 C) 0.7419 D) 0.7635 E) 0.7851

$$\int_0^1 K x^{2.2}\,dx = \left[\frac{K}{3.2} x^{3.2}\right]_0^1 = \frac{K}{3.2} = 1 \Rightarrow K = 3.2$$

$$F(x) = \int_0^x 3.2\, t^{2.2}\,dt = t^{3.2}\Big]_0^x = x^{3.2}$$

$$F(x) = 0.35 \Rightarrow x^{3.2} = 0.35 \Rightarrow x = \boxed{0.7203}$$

2. Let *X* be a continuous random variable with probability density function given by $f(x) = \dfrac{5x}{\left(2.5+x^2\right)^2}$ for $x \ge 0$, and 0 otherwise. Find the median of *X*.

A) 1.5811 B) 1.5337 C) 1.6286 D) 1.6760 E) 1.7234

$u = 2.5 + x^2$
$du = 2x\,dx$

$$\int \frac{5x}{(2.5+x^2)^2}\,dx = \int \frac{2.5}{u^2}\,du = -\frac{2.5}{u} + C = -\frac{2.5}{2.5+x^2} + C$$

$$F(x) = \int_0^x \frac{5t}{(2.5+t^2)^2}\,dt = -\left[\frac{2.5}{(2.5+t^2)}\right]_0^x = 1 - \frac{2.5}{2.5+x^2}$$

$$F(x) = 0.5 \Rightarrow 1 - \frac{2.5}{2.5+x^2} = 0.5 \Rightarrow \frac{2.5}{2.5+x^2} = 0.5 \Rightarrow x = \boxed{1.5811}$$

3. Let *X* be a continuous random variable with probability density function given by $f(x) = \dfrac{13.2x}{\left(6.6+x^2\right)^2}$ for $x \ge 0$, and 0 otherwise. Find the mode of *X*.

A) 1.4832 B) 1.5277 C) 1.5722 D) 1.6167 E) 1.6612

$$f'(x) = \frac{(6.6+x^2)^2(13.2) - (13.2x)(2)(6.6+x^2)(2x)}{(6.6+x^2)^4} = \frac{(6.6+x^2)(13.2) - 52.8x^2}{(6.6+x^2)^3}$$

$$f'(x) = 0 \Rightarrow (6.6+x^2)(13.2) - 52.8x^2 = 0$$

$$\Rightarrow 39.6x^2 = 87.12$$

$$\Rightarrow x = \boxed{1.4832}$$

4. Let X be a continuous random variable with cumulative distribution function given by $F(x) = 1 - e^{-0.491x}(1 + 0.491x)$ if $x \geq 0$, and 0 otherwise. Find the mode of X.

$\boxed{\text{A)}}$ 2.0367 B) 1.9145 C) 2.1589 D) 2.2811 E) 2.4033

$$f(x) = F'(x) = 0.491\, e^{-0.491x}(1+0.491x) - 0.491\, e^{-0.491x}$$

$$f(x) = 0.491\, x\, e^{-0.491x}$$

$$f'(x) = 0.491\, e^{-0.491x} - (0.491)^2\, x\, e^{-0.491x}$$

$$= 0.491 e^{-0.491x}\left[1 - 0.491\, x\right]$$

$$f'(x) = 0 \Rightarrow 1 - 0.491\, x = 0$$

$$\Rightarrow x = \boxed{2.0367}$$

5. Let X be a continuous random variable with survival function given by $S(x) = \dfrac{e^{-x}}{6 + e^{-x}}$ for $-\infty < x < \infty$. Find the 65th-percentile of X, $\pi_{0.65}$.

$\boxed{\text{A)}}$ −1.1727 B) −1.2431 C) −1.3134 D) −1.3838 E) −1.4542

$$P[X < x] = 0.65 \Rightarrow F(x) = 0.65 \Rightarrow S(x) = 0.35$$

$$S(x) = 0.35 \Rightarrow \frac{e^{-x}}{6 + e^{-x}} = 0.35$$

$$\Rightarrow e^{-x} = 2.1 + 0.35\, e^{-x}$$

$$\Rightarrow 0.65\, e^{-x} = 2.1$$

$$\Rightarrow x = \boxed{-1.1727}$$