HW 4.6(a) Key

1. Let X be a continuous random variable that follows a gamma distribution with a mean of 28 and a variance of 196. Calculate $P[X < 18.2]$.

   $\boxed{\text{A)}}$ 0.2640 B) 0.2157 C) 0.2318 D) 0.2479 E) 0.2801

$\mu = \alpha\beta = 28 \qquad \sigma^2 = \alpha\beta^2 = 196 \quad \Rightarrow \quad \beta = 7,\ \alpha = 4$

$f(x) = \frac{1}{7^4(6)} x^3 e^{-x/7}$

$P[X < 18.2] = \int_0^{18.2} f(x)\,dx$

$= \left[e^{-x/7}\left(\frac{x^3}{6\cdot 7^3} + \frac{x^2}{2\cdot 7^2} + \frac{x}{7} + 1\right)\right]_{18.2}^{0}$

$= 1 - e^{-2.6}(2.92933 + 3.38 + 2.6 + 1)$

$= \boxed{0.2640}$

		$\frac{1}{7^4} e^{-x/7}$
+	$\frac{1}{6}x^3$	$-\frac{1}{7^3} e^{-x/7}$
−	$\frac{1}{2}x^2$	$\frac{1}{7^2} e^{-x/7}$
+	x	$-\frac{1}{7} e^{-x/7}$
−	1	$e^{-x/7}$
+	0	*

2. Let X be a continuous random variable that follows a gamma distribution with a mean of 18 and a variance of 108. Calculate $P[X > 33.6 | X > 18.6]$.

   $\boxed{\text{A)}}$ 0.2054 B) 0.1678 C) 0.1803 D) 0.1928 E) 0.2179

$\mu = \alpha\beta = 18,\quad \sigma^2 = 108 \quad \Rightarrow \quad \beta = 6,\ \alpha = 3$

$f(x) = \frac{1}{6^3\cdot 2} x^2 e^{-x/6}$

$S(x) = \int_x^{\infty} f(t)\,dt = \left[e^{-t/6}\left(\frac{t^2}{2\cdot 6^2} + \frac{t}{6} + 1\right)\right]_{\infty}^{x}$

$S(x) = e^{-x/6}\left(\frac{x^2}{72} + \frac{x}{6} + 1\right)$

		$\frac{1}{6^3} e^{-t/6}$
+	$\frac{1}{2}t^2$	$-\frac{1}{6^2} e^{-t/6}$
−	t	$\frac{1}{6} e^{-t/6}$
+	1	$-e^{-t/6}$
−	0	*

$S(33.6) = 0.0823884$

$S(18.6) = 0.401163$

$P[X > 33.6 | X > 18.6] = \frac{P[X > 33.6]}{P[X > 18.6]} = \frac{S(33.6)}{S(18.6)} = \boxed{0.2054}$

3. Let X be a continuous random variable with probability density function given by $f(x) = k\,x^{2.2}\,e^{-kx}$. Given that the mean of X is 16.96, find $Var[X]$.

$\boxed{\text{A)}}$ 89.888 B) 73.438 C) 78.922 D) 84.405 E) 95.371

$X \sim GAM\left(\alpha = 3.2,\ \beta = \frac{1}{k}\right)$

$\mu = 3.2\beta = 16.96 \Rightarrow \beta = 5.3$

$\sigma^2 = 3.2(5.3)^2 = \boxed{89.888}$

4. Let X be a continuous random variable that follows a gamma distribution with $\alpha = 2$. Given that $P[X > 17.98] = 4.1e^{-3.1}$, find $E[X]$.

$\boxed{\text{A)}}$ 11.6 B) 10.2 C) 10.9 D) 12.3 E) 13.0

$P[X > 17.98] = \int_{17.98}^{\infty} \frac{1}{\beta^2}\, x e^{-x/\beta}\, dx$

$= e^{-x/\beta}\left[\frac{x}{\beta} + 1\right]_{\infty}^{17.98}$

$= e^{-17.98/\beta}\left[\frac{17.98}{\beta} + 1\right] = 4.1e^{-3.1}$

		$\frac{1}{\beta^2} e^{-x/\beta}$
+	x	$-\frac{1}{\beta} e^{-x/\beta}$
−	1	$e^{-x/\beta}$
+	0	*

$e^{-17.98/\beta} = e^{-3.1} \Rightarrow \beta = 5.8$ Check: $\frac{17.58}{5.8} + 1 = 4.1$ ✓

$E[X] = 2(5.8) = \boxed{11.6}$

5. Let X be a continuous random variable with probability density function given by $f(x) = 156.645x^n e^{-kx}$, where n is a positive integer. Given that the mean of X is 1.76, find $Var[X]$.

$\boxed{\text{A)}}$ 0.2816 B) 0.2301 C) 0.2472 D) 0.2644 E) 0.2988

$X \sim GAM\left(\alpha = n+1,\ \beta = -\frac{1}{k}\right)$

$\mu = \alpha\beta = 1.76 \Rightarrow \beta = \frac{1.76}{\alpha}$

$156.645 = \frac{1}{\left(\frac{1.76}{\alpha}\right)^{\alpha}(\alpha-1)!} \Rightarrow \frac{\alpha^{\alpha}}{1.76^{\alpha}(\alpha-1)!} = 156.645$

Using "table" fn: $\alpha = 11$, $\beta = 0.16$

$Var[X] = 11(0.16)^2 = \boxed{0.2816}$