HW 5.3(b) Key

1. Let X and Y be continuous random variables with joint probability density function given by $f(x,y)=\frac{1}{1782}\left(x+y^2\right)$ for $0 \le x \le 3$ and $0 \le y \le 12$. Find $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_0^{12} \frac{1}{1782}(x+y^2)\,dy = \frac{1}{1782}\left[xy + \frac{1}{3}y^3\right]_0^{12}$$

$$= \frac{1}{1782}\left[12x + 576\right]$$

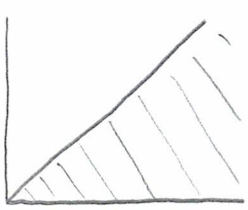
$$\boxed{f_X(x) = \frac{2}{297}x + \frac{32}{99}}$$

$$f_Y(y) = \int_0^{3} \frac{1}{1782}(x+y^2)\,dx = \frac{1}{1782}\left[\frac{1}{2}x^2 + xy^2\right]_0^{3}$$

$$= \frac{1}{1782}\left[\frac{9}{2} + 3y^2\right]$$

$$\boxed{f_Y(y) = \frac{1}{594}y^2 + \frac{1}{396}}$$

2. Let X and Y be continuous random variables with joint probability density function given by $f(x,y)=240e^{-(10x+14y)}$ for $0<y<x<\infty$. Find $f_X(x)$ and $f_Y(y)$.



$$f_X(x) = \int_0^{x} 240e^{-10x}e^{-14y}\,dy = -\frac{120}{7}e^{-10x}e^{-14y}\Big|_0^{x}$$

$$= -\frac{120}{7}\left[e^{-24x} - e^{-10x}\right]$$

$$\boxed{f_X(x) = \frac{120}{7}\left[e^{-10x} - e^{-24x}\right]}$$

$$f_Y(y) = \int_y^{\infty} 240e^{-10x}e^{-14y}\,dy = -24e^{-10x}e^{-14y}\Big|_y^{\infty}$$

$$= -24\left[0 - e^{-24y}\right]$$

$$\boxed{f_Y(y) = 24e^{-24y}}$$

3. Let X and Y be continuous random variables with joint probability density function given by $f(x,y)=\frac{6}{71}\left(\frac{13}{x^2y^4}+\frac{15}{x^3y^2}\right)$ for $x,y\geq 1$. Find $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \frac{6}{71}\int_1^\infty\left(\frac{13}{x^2y^4}+\frac{15}{x^3y^2}\right)dy = \frac{6}{71}\left[-\frac{1}{3}\frac{13}{x^2y^3}-\frac{1}{1}\frac{15}{x^3y}\right]_1^\infty$$

$$= \frac{6}{71}\left[\frac{13}{3x^2}+\frac{15}{x^3}\right]$$

$$\boxed{f_X(x) = \frac{26}{71x^2}+\frac{90}{71x^3}}$$

$$f_Y(y) = \frac{6}{71}\int_1^\infty\left(\frac{13}{x^2y^4}+\frac{15}{x^3y^2}\right)dx = \frac{6}{71}\left[-\frac{1}{1}\frac{13}{xy^4}-\frac{1}{2}\frac{15}{x^2y^2}\right]_1^\infty$$

$$= \frac{6}{71}\left[\frac{13}{y^4}+\frac{15}{2y^2}\right]$$

$$\boxed{f_Y(y) = \frac{78}{71y^4}+\frac{45}{71y^2}}$$

4. Let X and Y be continuous random variables with joint cumulative distribution function given by:

$F(x,y)=1-\frac{1}{2}e^{12(1-x)}-\frac{1}{2}e^{4(1-y)}-\frac{1}{2x^2}+\frac{e^{4(1-y)}}{2x^2}-\frac{1}{2y}+\frac{e^{12(1-x)}}{2y}$ for $x,y\geq 1$.

Find $f_X(x)$ and $f_Y(y)$.

$$F_X(x) = F(x,\infty) = 1-\frac{1}{2}e^{12(1-x)}-\frac{1}{2x^2}$$

$$f_X(x) = F_X'(x) = -\frac{1}{2}(-12)e^{12(1-x)}-(-2)\frac{1}{2x^3}$$

$$\boxed{f_X(x) = 6e^{12(1-x)}+\frac{1}{x^3}}$$

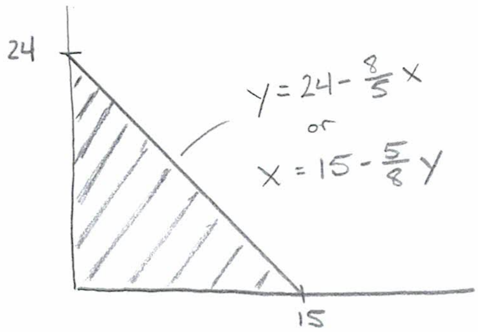
$$F_Y(y) = F(\infty,y) = 1-\frac{1}{2}e^{4(1-y)}-\frac{1}{2y}$$

$$f_Y(y) = F_Y'(y) = -\frac{1}{2}(-4)e^{4(1-y)}-(-1)\frac{1}{2y^2}$$

$$f_Y(y) = \boxed{2e^{4(1-y)}+\frac{1}{2y^2}}$$

5. Let X and Y be continuous random variables that are uniformly distributed on the region defined by $x \geq 0$, $y \geq 0$, and $8x + 5y \leq 120$.

Find $E[X]$ and $E[Y]$.



$$\text{Area} = \tfrac{1}{2}(24)(15) = 180$$

$$f(x,y) = \frac{1}{180}$$

$$f_X(x) = \int_0^{24-\frac{8}{5}x} \frac{1}{180}\, dy = \frac{2}{15} - \frac{2}{225}x$$

$$E[X] = \int_0^{15} \left(\frac{2}{15}x - \frac{2}{225}x^2\right) dx = \left[\frac{1}{15}x^2 - \frac{2}{675}x^3\right]_0^{15}$$

$$= 15 - 10 = \boxed{5}$$

$$f_Y(y) = \int_0^{15-\frac{5}{8}y} \frac{1}{180}\, dy = \frac{1}{12} - \frac{1}{288}y$$

$$E[Y] = \int_0^{24} \left(\frac{1}{12}y - \frac{1}{288}y^2\right) dy = \left[\frac{1}{24}y^2 - \frac{1}{864}y^3\right]_0^{24}$$

$$= 24 - 16 = \boxed{8}$$