HW 5.4 Key

1. Let X and Y be continuous random variables with joint probability density function given by $f(x,y) = kx^2y$ for $0 \le x \le 5$ and $0 \le y \le 8$. Find $Var[X]$.

$\boxed{\text{A)}}$ 0.9375 B) 0.8625 C) 1.0125 D) 1.0875 E) 1.1625

$(X \perp Y) \Rightarrow f(x,y) = kx^2y = f_X(x) f_Y(y) \Rightarrow f_X(x) = k_1 x^2$

$\int_0^5 k_1 x^2 \, dx = k_1 \left(\frac{1}{3}\right)\left[x^3\right]_0^5 = k\left(\frac{125}{3}\right) = 1 \Rightarrow k = \frac{3}{125}$

$E[X] = \int_0^5 \frac{3}{125} x^3 \, dx = \frac{3}{4(125)} x^4 \Big|_0^5 = \frac{3 \cdot 5}{4} = \frac{15}{4}$

$E[X^2] = \int_0^5 \frac{3}{125} x^4 \, dx = \frac{3}{5(125)} x^5 \Big|_0^5 = 15$

$Var[X] = 15 - \left(\frac{15}{4}\right)^2 = \boxed{0.9375}$

2. Let X and Y be continuous random variables with joint probability density function given by $f(x,y) = \dfrac{243x^6e^{-3x}}{10y^9}$ for $x \ge 0$ and $y \ge 1$. Find $E\left[X^2Y\right]$.

$\boxed{\text{A)}}$ 7.1111 B) 7.8222 C) 8.5333 D) 9.2444 E) 9.9556

$(X \perp Y) \Rightarrow f_X(x) = k_1 x^6 e^{-3x} \qquad f_Y(y) = k_2 y^{-9}$

$X \sim GAM(\alpha = 7, \beta = 3) \Rightarrow k_1 = \frac{3^7}{6!} = \frac{243}{80} \Rightarrow k_2 = 8$

$E[X] = \frac{7}{3}$, $Var[X] = \frac{7}{9} \Rightarrow E[X^2] = \frac{56}{9}$

$E[Y] = \int_1^\infty 8y^{-8} \, dy = \left[\frac{8}{7} y^{-7}\right]_\infty^1 = \frac{8}{7}$

$(X \perp Y) \Rightarrow E[X^2Y] = E[X^2]E[Y] = \frac{56}{9} \frac{8}{7} = \boxed{7.1111}$

3. Assume that X and Y are independent continuous random variables. Suppose that X in uniformly distributed on the interval $[1,5]$ and Y is uniformly distributed on the interval $[1,13]$. Find $E\left[(x+y)^{-2}\right]$.

$\boxed{\text{A)}}$ 0.01765 B) 0.01236 C) 0.01412 D) 0.01589 E) 0.01942

$f_X(x) = \frac{1}{4}$ $\quad f_Y(y) = \frac{1}{12}$ $\quad f(x,y) = \frac{1}{48}$

$E[(x+y)^{-2}] = \int_1^5 \int_1^{13} (x+y)^{-2} \frac{1}{48}\, dy\, dx$

$= \frac{1}{48} \int_1^5 \left[-(x+y)^{-1}\right]_1^{13} dx = \frac{1}{48} \int_1^5 \left[(x+1)^{-1} - (x+13)^{-1}\right] dx$

$= \frac{1}{48} \left[\ln(x+1) - \ln(x+13)\right]_1^5 = \frac{1}{48}\left[\ln 6 - \ln 18 - \ln 2 + \ln 14\right]$

$= \boxed{0.01765}$

4. Assume that X, Y, and Z are pairwise independent. In other ~~wise~~ words, each of the three random variables is independent with each of the other two. Given that $E[XY]=532$, $E[XZ]=266$, and $E[YZ]=392$, find $E[Z]$.

$\boxed{\text{A)}}$ 14 B) 11 C) 13 D) 15 E) 17

Since $X \perp Y$, $E[XY] = E[X]\,E[Y]$. (And so on.)

$$\left.\begin{aligned} E[X]E[Y] &= 532 \\ E[X]E[Z] &= 266 \end{aligned}\right\} \rightarrow \frac{E[Y]}{E[Z]} = \frac{532}{266} \Rightarrow E[Y] = 2\,E[Z]$$

$E[Y]\,E[Z] = 392 \Rightarrow 2(E[Z])^2 = 392 \Rightarrow E[Z] = \boxed{14}$

5. Let X and Y be continuous random variables with joint probability density function given by $f(x,y)=\frac{4}{203}(xy+14x+cy+d)$. Assuming that X and Y are independent, find d.

$\boxed{\text{A)}}$ 42   B) 36   C) 38   D) 40   E) 44

$(X \perp Y) \Rightarrow f(x,y) = f_X(x) f_Y(y) = [k_1(x+a)][k_2(y+b)]$

$f(x,y) = k_1k_2(xy + bx + ay + ab) = \frac{4}{203}(xy + 14x + cy + d)$

$\Rightarrow b = 14, \quad a = c, \quad d = ab = 14c, \quad k_1k_2 = \frac{4}{203}$

We need to find a (and thus c).

$\int_0^1 k_2(y+14)\,dy = k_2\left[\frac{1}{2}y^2 + 14y\right]_0^1 = \frac{29}{2}k_2 \overset{=1}{} \Rightarrow k_2 = \frac{2}{29}$

$\Rightarrow k_1 = \frac{2}{7}$

$\int_0^1 \frac{2}{7}(x+a)\,dx = \frac{2}{7}\left[\frac{1}{2}x^2 + ax\right]_0^1 = \frac{2}{7}\left(\frac{1}{2}+a\right) = 1$

$\Rightarrow \frac{1}{2} + a = \frac{7}{2} \Rightarrow a = 3$

$d = 3(14) = \boxed{42}$