HW 6.4 Key

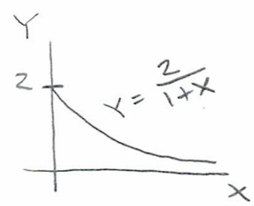
1. Let X be a continuous random variable with probability density function given by $f_X(x) = \frac{1}{156}(14+4x)$ for $0 \le x \le 6$. Let $Y = X^2$. Find the marginal pdf for Y.

$y = x^2 \Rightarrow x = \sqrt{y} \Rightarrow \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

$f_Y(y) = \frac{1}{156}(14 + 4\sqrt{y})\frac{1}{2\sqrt{y}} = \frac{1}{156}\left(\frac{7}{\sqrt{y}} + 2\right)$

$\boxed{f_Y(y) = \frac{7}{156\sqrt{y}} + \frac{1}{78}, \quad 0 \le y \le 36}$

2. Let X be a continuous random variable following a gamma distribution with $\alpha = 3$ and $\beta = 2.7$. Let $Y = \frac{2}{1+X}$. Find the marginal pdf for Y.



$y = 2/(1+x) \Rightarrow x = \frac{2}{y} - 1 = \frac{2-y}{y} \Rightarrow \frac{dx}{dy} = -\frac{2}{y^2}$

$x \ge 0 \Rightarrow y \in [0, 2]$

$f_X(x) = 9.8415\, x^2 e^{-2.7x}$

$f_Y(y) = 9.8415\left(\frac{2-y}{y}\right)^2 e^{-2.7(2-y)/y}\left(\frac{2}{y^2}\right)$

$\boxed{f_Y(y) = 19.683\frac{(2-y)^2}{y^4} e^{-2.7(2-y)/y}, \quad 0 \le y \le 2}$

3. Let X be a continuous random variable with probability density function given by $f_X(x) = 3x^2$ for $0 \le x \le 1$. Let $Y = 16 + 4X$. Find the marginal pdf for Y.

$y = 16 + 4x \Rightarrow x = \frac{1}{4}y - 4 \Rightarrow \frac{dx}{dy} = \frac{1}{4}$

$0 \le x \le 1 \Rightarrow 16 \le y \le 20$

$f_Y(y) = 3\left(\frac{1}{4}y - 4\right)^2 \frac{1}{4} = 3\left(\frac{1}{4}\right)^2 (y-16)^2 \left(\frac{1}{4}\right)$

$\boxed{f_Y(y) = \frac{3(y-16)^2}{4^3}, \quad 16 \le y \le 20}$

4. Let X be a continuous random variable with probability density function given by $f_X(x) = \frac{1.3}{x^{2.3}}$ for $x \geq 1$.

Let $Y = \frac{3X+1}{X}$. Find the marginal pdf for Y.

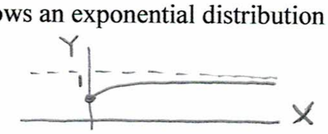
$$y = 3 + \frac{1}{x} \Rightarrow x = \frac{1}{y-3} \Rightarrow \frac{dx}{dy} = -\frac{1}{(y-3)^2}$$

$$x \geq 1 \Rightarrow y \in [3, 4]$$

$$f_Y(y) = \frac{1.3}{\left(\frac{1}{y-3}\right)^{2.3}} \frac{1}{(y-3)^2} = 1.3\,(y-3)^{2.3} \frac{1}{(y-3)^2}$$

$$\boxed{f_Y(y) = 1.3\,(y-3)^{0.3}, \quad 3 \leq y \leq 4}$$

5. Let X be a continuous random variable that follows an exponential distribution with a mean of $1/26$. Let $Y = \frac{1}{1+e^{-2X}}$. Find the marginal pdf for Y.



$$y = \frac{1}{1+e^{-2x}} \Rightarrow x = -\frac{1}{2}\ln\left(\frac{1}{y}-1\right) = -\frac{1}{2}\ln\left(\frac{1-y}{y}\right) = \frac{1}{2}\ln\left(\frac{y}{1-y}\right)$$

$$x = \frac{1}{2}\ln y - \frac{1}{2}\ln(1-y) \Rightarrow \frac{dx}{dy} = \frac{1}{2y} + \frac{1}{2(1-y)} = \frac{1}{2y(1-y)}$$

$$x \geq 0 \Rightarrow y \in [0.5, 1]$$

$$f_X(x) = 26\,e^{-26x}$$

$$f_Y(y) = 26\,e^{-26\left(\frac{1}{2}\right)\ln\left(\frac{y}{1-y}\right)} \frac{1}{2y(1-y)} = 13\,e^{\ln\left[\left(\frac{y}{1-y}\right)^{-13}\right]} \frac{1}{y(1-y)}$$

$$= 13\,\frac{(1-y)^{13}}{y^{13}} \frac{1}{y(1-y)}$$

$$\boxed{f_Y(y) = 13\,\frac{(1-y)^{12}}{y^{14}}, \quad 0.5 \leq y \leq 1}$$