Ch 01 – The Measurement of Interest

Simple Interest

$a(t) = 1 + i\,t$

Effective Rate of Interest

$i_t = \dfrac{a(t) - a(t-1)}{a(t-1)}$

$a(t) = (1+i_1)(1+i_2)\cdot\ldots\cdot(1+i_t)$

Effective Rate of Discount

$d_t = \dfrac{a(t) - a(t-1)}{a(t)}$

$a(t) = (1-d_1)^{-1}(1-d_2)^{-1}\cdot\ldots\cdot(1-d_t)^{-1}$

Force of Interest

Variable: $\delta_t = \dfrac{a'(t)}{a(t)}$, $\quad a(t) = \exp\left(\int_0^t \delta_r\,dr\right)$

Constant: $\delta = \ln(1+i)$, $\quad a(t) = e^{\delta t}$

AV at time t_2 of 1 paid at time t_1: $a(t1 \to t2) = \dfrac{a(t_2)}{a(t_1)}$

PV at time t_1 of 1 paid at time t_2: $\dfrac{1}{a(t1 \to t2)} = \dfrac{a(t_1)}{a(t_2)}$

Compound Interest

AV: $a(t) = (1+i)^t = (1-d)^{-t} = v^{-t} = e^{\delta t}$

PV: $\dfrac{1}{a(t)} = (1+i)^{-t} = (1-d)^t = v^t = e^{-\delta t}$

Identities:

$v = \dfrac{1}{1+i} = 1-d \qquad i = \dfrac{d}{1-d}$

$d = \dfrac{i}{1+i} = iv \qquad i - d = i\,d$

Nominal Rates

$i^{(m)}$ = nominal annual interest rate, compounded m-thly.

$j = \dfrac{i^{(m)}}{m}$ = effective m-thly interest rate.

$a(t) = (1+i)^t = (1+j)^{mt}$

Converting Rates: $1 + i = (1+j)^m$

$d^{(m)}$ = nominal annual discount rate, compounded m-thly.

$a(t) = (1-d)^{-t} = \left(1 - \dfrac{d^{(m)}}{m}\right)^{-mt}$

Ch 03 – Basic Annuities

Finite Geometric Series

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r} = \frac{\text{First Term} - \text{First Ommitted Term}}{1 - \text{Common Ratio}}$$

Annuity-Immediate (Payment at end of year)

PV: $a_{\overline{n}|i} = v + v^2 + \ldots + v^n$, $\quad a_{\overline{n}|i} = \dfrac{1-v^n}{i}$

AV: $s_{\overline{n}|i} = 1 + (1+i) + \ldots + (1+i)^{n-1}$, $\quad s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$

Identities:

$s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i} \qquad \dfrac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n \qquad \dfrac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n}$

Annuity-Due (Payment at beginning of year)

PV: $\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + \ldots + v^{n-1}$, $\quad \ddot{a}_{\overline{n}|i} = \dfrac{1-v^n}{d}$

AV: $\ddot{s}_{\overline{n}|i} = (1+i) + (1+i)^2 + \ldots + (1+i)^n$, $\quad \ddot{s}_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{d}$

Identities:

$\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i} = a_{\overline{n-1}|i} + 1$

$\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i} = s_{\overline{n+1}|i} - 1$

Deferred Annuity

PV of an annuity with first payment at $t = m+1$:

- ${}_{m|}a_{\overline{n}|i} = v^m a_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{n}|i}$

Perpetuities

$a_{\overline{\infty}|i} = \dfrac{1}{i} \qquad \ddot{a}_{\overline{\infty}|i} = \dfrac{1}{d} = 1 + a_{\overline{\infty}|i}$

Ch 12 – Swaps

Interest Rate Swaps

- Non-level sequence of payments represent interest payment based on varying spot rates.
- Goal is to find equivalent fixed interest rate.
- Consider a three year period with spot rates s_1, s_2, and s_3. The fixed swap rate R is given by:

$$1 = \frac{R}{1+s_1} + \frac{R}{(1+s_2)^2} + \frac{1+R}{(1+s_3)^3}$$

Ch 04 – More General Annuities

Fission

PV of 1 at the end of every n years for kn years: $\dfrac{a_{\overline{kn}|i}}{s_{\overline{n}|i}}$

PV of 1 at the beg of every n years for kn years: $\dfrac{a_{\overline{kn}|i}}{a_{\overline{n}|i}}$

Palindromic Annuities

PV of 1, 2, 3, ..., n - 1, n, n - 1, ..., 3, 2, 1 (paid at end of year) is: $PV = a_{\overline{n}|i} \cdot \ddot{a}_{\overline{n}|i}$.

Arithmetic Annuities

P = payment at $t=1$, Q = annual increase in payment

- $PV = P\,a_{\overline{n}|} + Q\,\dfrac{a_{\overline{n}|} - n\,v^n}{i}$
- $AV = P\,s_{\overline{n}|} + Q\,\dfrac{s_{\overline{n}|} - n}{i}$

Special Case: $P = Q = 1$

$(Ia)_{\overline{n}|} = \dfrac{\ddot{a}_{\overline{n}|} - n\,v^n}{i}$, $(Is)_{\overline{n}|} = \dfrac{\ddot{s}_{\overline{n}|} - n}{i}$

Special Case: $P = n$, $Q = -1$

$(Da)_{\overline{n}|} = \dfrac{n - a_{\overline{n}|}}{i}$, $(Ds)_{\overline{n}|} = \dfrac{n(1+i)^n - s_{\overline{n}|}}{i}$

Increasing Perpetuity: $(Ia)_{\overline{\infty}|} = \dfrac{1}{i\,d} = \dfrac{1}{i} + \dfrac{1}{i^2}$

Increasing to level Perpetuity: $PV = \dfrac{\ddot{a}_{\overline{n}|i}}{i}$

Geometric Annuities (Pmts inc at a rate of *k*)

Adjusted Interest Rate: $i' = \dfrac{i-k}{1+k}$

Adjusted Present Value Factor: $v' = \dfrac{1+k}{1+i}$

$PV = v\,\ddot{a}_{\overline{n}|i'} = \dfrac{1}{1+k}\,a_{\overline{n}|i'}$

Continuous Annuities

AV of 1 paid continuously over a year: $\bar{s}_{\overline{1}|} = \dfrac{i}{\delta}$

PV of payments made continuously for n years:

Rate	FoI	PV
1	δ	$\bar{a}_{\overline{n}|} = \int_0^n v^t\,dt = \dfrac{1 - v^n}{\delta} = \dfrac{i}{\delta}\,a_{\overline{n}|}$
t	δ	$(\bar{I}\,\bar{a})_{\overline{n}|} = \int_0^n t\,v^t\,dt = \dfrac{\bar{a}_{\overline{n}|} - n\,v^n}{\delta}$
$f(t)$	δ	$\int_0^n f(t)\,v^t\,dt$
$f(t)$	δ_t	$\int_0^n \dfrac{f(t)}{a(t)}\,dt$

Ch 05 – Yield Rates

NPV and IRR

Two methods for comparing investments:

- *Net Present Value (NPV)*: Sum the PV of cash inflows and cash outflows. Choose investment with greatest NPV.
- *Internal Rate of Return (IRR)*: The rate such that the PV of cash inflows is equal to PV of cash outflows. Choose investment with greatest IRR.

DWR and TWR (Example)

	$t=0$	$t=1/4$	$t=1/2$	$t=1$
Beg:	0	120	90	220
Trans	+ 100	− 40	+ 110	− 220
End:	100	80	200	0

$$\text{DWR} = \frac{-100 + 40 - 110 + 220}{100(1) - 40(3/4) + 110(1/2)} = 0.4 = 40\%$$

$$\text{TWR} = \frac{120}{100}\,\frac{90}{80}\,\frac{220}{200} - 1 = 0.485 = 48.5\%$$

Ch 06 – Amortization

Notation

- L = original loan amount.
- R = annual payment.
- B_t = unpaid balance at time t. Note that $B_0 = L$.
- I_t = interest portion of payment t.
- P_t = principal portion of payment t.

B_t - Prospective & Retrospective Methods

- *Prospective*: PV of all future payments: $B_t = R\,a_{\overline{n-t}|}$
- *Retrospective*: AV of loan amount minus AV of all past payments: $B_t = L(1+i)^t - R\,s_{\overline{t}|}$

Amortization Formulas

- $R = L\,/\,a_{\overline{n}|}$
- $B_t = R\,a_{\overline{n-t}|} = L(1+i)^t - R\,s_{\overline{t}|}$
- $I_t = R\left[1 - v^{n-t+1}\right] = i\,B_{t-1}$
- $P_t = R\,v^{n-t+1} = R - I_t$

Sinking Funds

- $I = i\,L$ (interest payment on loan)
- $SFD = L\,/\,s_{\overline{n}|j}$ (sinking fund deposit)
- Total Payment $= I + SFD$
- Net Interest Paid $= I -$ Interest earned by SF
- Net Balance $= L -$ Balance of SF

Ch 07 – Bonds

Notation

- P = price of bond
- F = par value (or face value)
- r = coupon rate per payment period
- Fr = amount of each coupon
- C = redemption amount of bond
- i = interest (yield) rate per pmt period
- n = number of payments
- g = special coupon rate $(Fr = Cg)$

Bond Pricing Formulas

- Basic Formula: $P = Fr\,a_{\overline{n|}i} + Cv^n$
- P/D Formula: $P = C + (Fr - Ci)a_{\overline{n|}i}$

Premium vs. Discount

	Premium	Discount
Condition	$P > C$ $g > i$	$P < C$ $g < i$
Amortization Process	Write-Down of Premium	Write-Up of Discount
Book Value	Decreases over time	Increases over time
P_t Formula	$(Fr - Ci)v^{n-t+1}$	$(Ci - Fr)v^{n-t+1}$

Price Between Coupon Dates

Price k time units after last coupon:

- *Full Price* (*Dirty Price*): Actual purchase price. Includes PV of the next coupon to be paid: $B_{t+k} = B_t(1+i)^k$
- *Market Price* (*Clean Price*): Does not include PV of next coupon. Given by: $B_{t+k} - kFr = B_t(1+i)^k - kFr$

Callable Bonds

The highest price that will guarantee a yield rate of i is the lowest price over all possible redemption dates.

- *Premium Bond*: Earlier redemption dates are worse for the investor
- *Discount Bond*: Later redemption dates are worse for the investor

Ch 08 – Financial Instruments

Share pricing for Stocks

- Price of a preferred stock paying coupons of K:
  $P = \frac{K}{i}$
- The theoretical price of a common stock with expected dividend D and growth rate k is $P = \frac{D}{i-k}$

Ch 09 – Advanced Financial Analysis

Inflation

- i = effective (nominal) rate of interest
- r = inflation rate
- $i' = \frac{i-r}{1+r}$ = real rate of interest

Spot Rates and Forward Rates

- s_n = annual effective yield rate for an n-year investment.
- f_n = annual effective interest rate for one year period from time n to $n+1$.
- Finding forward rates given spot rates:
  $$1+f_n = \frac{(1+s_{n+1})^{n+1}}{(1+s_n)^n}$$
- Finding spot rates given forward rates:
  $(1+s_n)^n = (1+f_0)\cdot(1+f_1)\cdot\ldots\cdot(1+f_{n-1})$

Duration and Convexity

- $MacD = \frac{\sum(t\cdot v^t\cdot CF_t)}{\sum(v^t\cdot CF_t)} = -\frac{P'(\delta)}{P(\delta)}$
- $ModD = v\cdot MacD = -\frac{P'(i)}{P(i)}$
- $MacD = n$ for zero coupon bond
- $Conv = \frac{\sum[t(t+1)v^{t+2}CF_t]}{\sum(v^t\cdot CF_t)} = \frac{P''(i)}{P(i)}$

The change in the price of an asset or liability resulting from a change in i can be approx. by:

$$\Delta P \approx P(i)\cdot\left[-(\Delta i)ModD + \frac{1}{2}(\Delta i)^2(Conv)\right]$$

Duration of a Portfolio

Assume a portfolio contains n investments, each with present value PV_i and duration $ModD_i$. The duration of the portfolio is given by:

- $ModD = \frac{\sum(PV_i\cdot ModD_i)}{\sum PV_i}$

Immunization

Redington	Full
Protects against small Δi	Protects against all Δi
PV of assets = PV of liabilities	
Duration of assets = Duration of liabilities	
$Conv_A > Conv_L$ $(P''_A > P''_L)$	There is one asset cash flow before each liability cash outflow and one after it.

Exact Matching (Dedication)

An enterprise may be immunized by exactly matching the time and value of cash inflows with cash outflows.