01 – PUT-CALL PARITY

Forward Prices for Stocks

- The stock pays no dividends:
  - $F_{0,T} = S_0 e^{rT}$ and $F^P_{0,T} = S_0$
- The stock pays discrete dividends:
  - $F_{0,T} = S_0 e^{rT} - \text{AV}(\text{Divs})$ and $F^P_{0,T} = S_0 - \text{PV}(\text{Divs})$
- The stock pays continuous dividends:
  - $F_{0,T} = S_0 e^{(r-\delta)T}$ and $F^P_{0,T} = S_0 e^{-\delta T}$
- In any case: $F_{0,T} = PV\left(F^P_{0,T}\right)$

Forwards on Currency Exchanges

- $F_{0,T} = x_0 e^{(r_d - r_f)T}$ and $F^P_{0,T} = x_0 e^{-r_f T}$

Put-Call Parity

- $C(K, T) - P(K, T) = F^P_{0,T} - PV(K) = S_0 e^{-\delta T} - K e^{-rT}$
- Only applies to European Options.

02 – COMPARING OPTIONS

General Principles
- Premiums for ordinary options are never negative.
- A Eur. option is always cheaper than a similar American option.

Bounds for European Option Premiums
- $S e^{-\delta T} - K e^{-rT} \le C_{EUR} \le S e^{-\delta T}$
- $K e^{-rT} - S e^{-\delta T} \le P_{EUR} \le K e^{-rT}$

Bounds for American Option Premiums
- $S - K \le C_{AM} \le S$
- $K - S \le P_{AM} \le K$

Comparing Options with Different Expiration Dates

Assume $T_1 < T_2$.
- For American Calls or Puts: $V(S, K, T_1) \le V(S, K, T_2)$
- For European Calls or Puts: $V(S, K, T_1) \le V(S, K e^{-r(T_2 - T_1)}, T_2)$

Different Strike Prices

Assume $K_1 < K_2 < K_3$.

C is a decreasing function of *K* and $-1 \le C_K \le 0$.
- $C(S, K_1, T) > C(S, K_2, T)$
- $C_{AM}(K_1) - C_{AM}(K_2) < K_2 - K_1$
- $C_{EUR}(K_1) - C_{EUR}(K_2) < PV(K_2 - K_1)$

P is an increasing function of *K* and $0 \le P_K \le 1$.
- $P(S, K_1, T) < P(S, K_2, T)$
- $P_{AM}(K_2) - P_{AM}(K_1) < K_2 - K_1$
- $P_{EUR}(K_2) - P_{EUR}(K_1) < PV(K_2 - K_1)$

Convexity
- Let $a = \dfrac{K_3 - K_2}{K_3 - K_1}$ and $b = \dfrac{K_2 - K_1}{K_3 - K_1}$.
- Then $K_2 = a K_1 + b K_3$.
- By convexity, $V(K_2) \le a V(K_1) + b V(K_3)$.
- Equivalently: $\dfrac{V(K_1) - V(K_2)}{K_2 - K_1} > \dfrac{V(K_2) - V(K_3)}{K_3 - K_2}$

03 – ONE PERIOD BINOMIAL TREES

True Probabilities

- $E[S_h] = p \cdot S \cdot u + (1-p) \cdot S \cdot d$
- $\alpha = \dfrac{1}{h} \cdot \ln\left(\dfrac{e^{\delta h} E[S_h]}{S}\right)$, $\alpha = g + \delta$
- $C_u = \max[0, S u - K]$, $C_d = \max[0, S d - K]$
- $P_u = \max[0, K - S u]$, $P_d = \max[0, K - S d]$
- $E[PO] = p V_u + (1-p) V_d$
- $V = e^{-\gamma h} E[PO]$, but γ is generally not known.

Risk-Neutral Pricing

- $p^* = \dfrac{e^{(r-\delta)h} - d}{u - d}$
- $E[PO] = p^* V_u + (1 - p^*) V_d$
- $\text{Call}(K, h) = \left[p^* C_u + (1 - p^*) C_d\right] e^{-rh}$
- $\text{Put}(K, h) = \left[p^* P_u + (1 - p^*) P_d\right] e^{-rh}$

Replicating Portfolios

- $\Delta e^{\delta h} S u + B e^{rh} = V_u$, $\Delta e^{\delta h} S d + B e^{rh} = V_d$
- $\Delta = \left(\dfrac{V_u - V_d}{S u - S d}\right) e^{-\delta h}$, $B = \left(\dfrac{u V_d - d V_u}{u - d}\right) e^{-rh}$
- $V = \Delta S + B$
- For Calls: $\Delta \ge 0$ and $B \le 0$
- For Puts: $\Delta \le 0$ and $B \ge 0$
- $\Delta_C - \Delta_P = e^{-\delta h}$

Volatility

- Annual volatility σ is given by $\sigma^2 = \dfrac{1}{h} \cdot \text{Var}\left[\ln\left(\dfrac{S_h}{S}\right)\right]$.

Binomial Tree Models
- Standard Model (Forward Tree)
  - $u = e^{(r-\delta)h + \sigma\sqrt{h}}$ and $d = e^{(r-\delta)h - \sigma\sqrt{h}}$
- Cox-Ross-Rubinstein Tree
  - $u = e^{\sigma\sqrt{h}}$ and $d = e^{-\sigma\sqrt{h}}$
- Lognormal Tree (Jarrow/Rudd Tree)
  - $u = e^{(r - \delta - 0.5\sigma^2)h + \sigma\sqrt{h}}$ and $d = e^{(r - \delta - 0.5\sigma^2)h - \sigma\sqrt{h}}$
- Each model above satisfies $u/d = e^{2\sigma\sqrt{h}}$

05 – UTILITY

Definitions
- $W_H = \dfrac{p^*}{p}$, $W_L = \dfrac{1 - p^*}{1 - p}$
- $U_H = \dfrac{1}{1+r} W_H$, $U_L = \dfrac{1}{1+r} W_L$
- $Q_H = p U_H$, $Q_L = (1-p) U_L$

Relationships
- $p W_H + (1-p) W_L = 1$
- $p U_H + (1-p) U_L = \dfrac{1}{1+r}$
- $Q_H + Q_L = \dfrac{1}{1+r}$

Pricing with Utility
- $S = [Q_H S_u + Q_L S_d](1 + \delta)$, $V = Q_H V_u + Q_L V_d$

07 – LOGNORMAL STOCK MODEL

The Lognormal Distribution

- $X \sim \text{Normal}(m, v^2)$, $Y = e^X$
- $E[Y] = e^{m + 0.5v^2}$
- $\text{Var}[Y] = (E[Y])^2[e^{v^2} - 1] = e^{2m + v^2}(e^{v^2} - 1)$
- $\text{Med}[Y] = e^m$
- $\text{Mode}[Y] = e^{m - v^2}$

The Lognormal Stock Model

- $S_t = S_0 e^{R_t}$, $R_t \sim \text{Normal}(m, v^2)$
- $m = (\alpha - \delta - 0.5\sigma^2)t$, $v = \sigma\sqrt{t}$
- $E[S_t] = S_0 e^{(\alpha - \delta)t}$
- $Med[S_t] = S_0 e^m$

Methods of Stating Volatility

- $\text{Var}[\ln(F^P_{0,T})] = \sigma^2 T$ or $\text{Var}[\ln(F_{0,T})] = \sigma^2 T$
- $\ln\left[\dfrac{E[S_t]}{\text{Med}[S_t]}\right] = 0.5\sigma^2 t$

Prediction Intervals

- $Z_{p/2} = N(1-p/2)$
- $(1-p)$ Confidence Interval for At : $(m - z_{p/2}v\,,\ m + z_{p/2}v)$
- $(1-p)$ Prediction Interval for S_t : $(S_0 e^{m - z_{p/2}v},\ S_0 e^{m + z_{p/2}v})$

Conditional Payoffs (Using True Probabilities)

Probability of Option Payoff

- $\hat{d}_1 = \dfrac{\ln(S_0 / K) + (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}}$
- $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{t}$
- $Pr[S_t < K] = N(-\hat{d}_2)$
- $Pr[S_t > K] = N(\hat{d}_2)$

Partial and Conditional Expectations

- $PE[S_t \mid S_t < K] = E[S_t]N(-\hat{d}_1)$
- $PE[S_t \mid S_t > K] = E[S_t]N(\hat{d}_1)$
- $E[S_t \mid S_t < K] = \dfrac{S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)}{N(-\hat{d}_2)}$
- $E[S_t \mid S_t > K] = \dfrac{S_0 e^{(\alpha-\delta)t} N(\hat{d}_1)}{N(\hat{d}_2)}$

Expected Payoff

- $E[\text{Call PO}] = S_0 e^{(\alpha-\delta)t} N(\hat{d}_1) - K\,N(\hat{d}_2)$
- $E[\text{Put PO}] = K\,N(-\hat{d}_2) - S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$

08 – ESTIMATING LOGNORMAL PARAMETERS

See Chapter 11.

09 – BLACK-SCHOLES FORMULA

General Black-Scholes Formula

- $d_1 = \dfrac{\ln(F^P(S) / F^P(K)) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$
- $d_2 = \dfrac{\ln(F^P(S) / F^P(K)) - 0.5\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$
- $C = F^P(S)N(d_1) - F^P(K)N(d_2)$
- $P = F^P(K)N(-d_2) - F^P(S)N(-d_1)$

Black-Scholes Formula for Standard Options

- $d_1 = \dfrac{\ln(S_0 / K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}}$
- $d_2 = \dfrac{\ln(S_0 / K) + (r - \delta - 0.5\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$
- $C = S_0 e^{-\delta t} N(d_1) - K e^{-rt} N(d_2)$
- $P = K e^{-rt} N(-d_2) - S_0 e^{-\delta t} N(-d_1)$

Black-Scholes Formula for Currency Options

- $r = r_d$ is the domestic risk free rate
- $\delta = r_f$ is the foreign risk free rate
- $S = x_0$ is the current exchange rate
- $d_1 = \dfrac{\ln(x_0 / K) + (r_d - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}}$
- $d_2 = \dfrac{\ln(x_0 / K) + (r_d - r_f - 0.5\sigma^2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$
- $C = x_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2)$
- $P = K e^{-r_d T} N(-d_2) - x_0 e^{-r_f T} N(-d_1)$

Black-Scholes Formula for Futures Options

- $d_1 = \dfrac{\ln(F / K) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$
- $C = F e^{-rT} N(d_1) - K e^{-rT} N(d_2)$
- $P = K e^{-rT} N(-d_2) - F e^{-rT} N(-d_1)$

Black Formula for Bond Options

- Asset is bond worth \$1 at time $T + S$.
- Option expires at time T.
- $F = P_0(T, T+S) = \dfrac{P(0, T+S)}{P(0,T)}$
- $d_1 = \dfrac{\ln(F / K) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$
- $C = P(0,T)[F\,N(d_1) - K\,N(d_2)]$
- $P = P(0,T)[K\,N(-d_2) - F\,N(-d_1)]$

10 – THE GREEKS

Delta $(\Delta = V_S)$

- $0 \le \Delta_C \le 1$ and $-1 \le \Delta_P \le 0$
- $\Delta_C = e^{-\delta T} N(d_1)$ and $\Delta_P = -e^{-\delta T} N(-d_1)$
- $\Delta_C - \Delta_P = e^{-\delta T}$

Gamma $(\Gamma = V_{SS})$

- $\Gamma_C = \Gamma_P$
- $\Gamma \ge 0$

Vega (V_σ)

- $\text{vega}_C = \text{vega}_P$
- $\text{vega} \ge 0$

Theta $(\theta = V_t)$

- $\theta_C - \theta_P = \delta S e^{-\delta T} - r K e^{-rT}$
- θ is *usually* negative

Rho $(\rho = V_r)$

- $\rho_C - \rho_P = T K e^{-rT}$
- $\rho_C \ge 0$ and $\rho_P \le 0$

Psi $(\psi = V_\delta)$

- $\psi_C - \psi_P = -T S e^{-\delta T}$
- $\psi_C \le 0$ and $\psi_P \ge 0$

Replicating Portfolios

- A call can be replicated by buying Δ_C shares and borrowing $K e^{-rT} N(d_2)$.
- A put can be replicated by selling $|\Delta_P|$ shares and lending $K e^{-rT} N(-d_2)$.

Greeks for Portfolios

- Let G be an arbitrary Greek.
- If $\pi = \sum A_i$, then $G_\pi = \sum G_i$.

Elasticity

- Elasticity: $\Omega = \dfrac{S\Delta}{V}$
- $\Omega_C \ge 1$ and $\Omega_P \le 0$
- $\sigma_{option} = \sigma_{stock} |\Omega|$
- $\gamma - r = \Omega(\alpha - r)$
- The elasticity of a portfolio is the price-weighted average of the elasticity of its instruments.

Sharpe Ratio

- $\phi_{stock} = \dfrac{\alpha - r}{\sigma_{stock}}$ and $\phi_{option} = \dfrac{\gamma - r}{\sigma_{option}}$
- $\phi_{call} = \phi_{stock}$ and $\phi_{put} = -\phi_{stock}$

11 – ESTIMATING VOLATILITY

Estimating Lognormal Parameters (Historical Volatility)

- Observed stock prices: $S_1, S_2, \ldots, S_n$
- Observed returns: $r_i = \ln(S_i / S_{i-1})$
- Standard Deviation (per period): $\hat{v} = \hat{\sigma}_h = \sqrt{\dfrac{1}{n-1} \sum (r_i - \bar{r})^2}$
- Mean (per period): $\hat{m} = \dfrac{1}{n} \sum r_i = \dfrac{1}{n} \ln\left(\dfrac{S_n}{S_0}\right)$
- Annual Volatility estimate (**Historical Volatility**): $\hat{\sigma} = \dfrac{\hat{\sigma}_h}{\sqrt{h}} = \dfrac{\hat{v}}{\sqrt{h}}$
- Annual Return estimate: $\hat{\alpha} = \dfrac{\hat{m}}{h} + \delta + 0.5\,\hat{\sigma}^2$
- Tip: Enter r_i's into TI-30XS to find $\hat{m}$ and $\hat{v}$ using 1-Var Stats.

Implied Volatility

- Assume S, K, T, r, δ, and V_0 are known.
- Implied volatility $\hat{\sigma}$ is the solution to $V_0 = V(S, K, \hat{\sigma}, r, T, \delta)$.
- Use implied volatility (rather than historical volatility) in the Black-Scholes formula.

12 – DELTA HEDGING

Overnight Profit

- Profit during period $[0, h]$ is given by: $[C(S_0) - \Delta S_0]e^{rh} + [\Delta e^{\delta h} S_h - C(S_h)]$
- Break-Even occurs at prices $S_0 \pm S_0 \sigma \sqrt{h}$.
- Market maker has positive profit if $S_0 - S_0 \sigma \sqrt{h} < S_h < S_0 + S_0 \sigma \sqrt{h}$

Delta-Gamma-Theta Approximation

- $V_{t+h} \approx V_t + \Delta \epsilon + 0.5\,\Gamma \epsilon^2 + h\theta$

This is an alternate form of Ito's Lemma:

- $dV = V_t\,dt + V_s\,dS + 0.5\,V_{SS}(dS)^2$

Greeks for Binomial Trees

- $\Delta(S,0) = \left[\dfrac{V_u - V_d}{Su - Sd}\right] e^{-\delta h}$
- $\Gamma(S,0) \approx \Gamma(S,h) = \dfrac{\Delta(Su,h) - \Delta(Sd,h)}{Su - Sd}$
- $V(Sud, 2h) = V(S,0) + \Delta(S,0)\epsilon + 0.5\Gamma(S,0)\epsilon^2 + 2h\,\theta(S,0)$

Rehedging (Boyle-Emanuel Formula)

- Assume portfolio is rehedged every h years.
- Periodic variance in return: $\text{Var}[r_h] = 0.5(S^2 \sigma^2 \Gamma h)^2$
- Annual variance in return: $0.5(S^2 \sigma^2 \Gamma)^2 h$

13, 14 – EXOTIC OPTIONS

Asian Options

Averages
- Arithmetic Average: $A(S) = \frac{1}{n}\sum S_i$
- Geometric Average: $G(S) = \sqrt[n]{\prod S_i}$
- $G(S) \le A(S)$

Asian Options
- Average Price Call: $PO = \max[0,\ \bar{S} - K]$
- Average Price Put: $PO = \max[0, K - \bar{S}]$
- Average Strike Call: $PO = \max[0,\ S - \bar{S}]$
- Average Strike Put: $PO = \max[0,\ \bar{S} - S]$

Barrier Options

Knock-In Options
- Up-and-in: $B > S_0$
- Down-and-in: $B < S_0$

Knock-Out Options
- Up-and-out: $B > S_0$
- Down-and-out: $B < S_0$

Relationship to Ordinary Options
- If $B \le K$, then an up-and-in call is equal to an ordinary call.
- If $B \ge K$, then a down-and-in put is equal to an ordinary put.
- Knock-In + Knock-Out = Ordinary Option

Compound Options

- CallOnCall: Option to buy a call.
- CallOnPut: Option to buy a put.
- PutOnCall: Option to sell a call.
- PutOnPut: Option to sell a put.

Parity Relations
- $CallOnCall - PutOnCall = Call - x\,e^{-rt_1}$
- $CallOnPut - PutOnPut = Put - x\,e^{-rt_1}$

Exchange Options

- S_t = price of underlying asset
- K_t = price of strike asset
- $\sigma = \sqrt{\sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K}$
- $d_1 = \dfrac{\ln(S_0 / K_0) + (\delta_K - \delta_S + 0.5\sigma^2)T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$
- $C = S_0 e^{-\delta_S T} N(d_1) - K_0 e^{-\delta_K T} N(d_2)$
- $P = K_0 e^{-\delta_K T} N(-d_2) - S_0 e^{-\delta_S T} N(-d_1)$

Relationships between Calls and Puts
- $\text{Call} - \text{Put} = S_0 e^{-\delta_S T} - K_0 e^{-\delta_K T}$
- $\text{Call}(S = A,\ K = B) = \text{Put}(S = B,\ K = A)$

Chooser Options

- $V_t = \max\left[C(S_t, K, T-t),\ P(S_t, K, T-t)\right]$
- $V_0 = C(S_0, K, T) + e^{-\delta(T-t)} P\left(S_0, K e^{-(r-\delta)(T-t)}, t\right)$

13, 14 – EXOTIC OPTIONS

All-Or-Nothing Options

Option	Price at time 0
$S \mid S > K$ (AONC)	$S_0 e^{-\delta T} N(d_1)$
$S \mid S < K$ (AONP)	$S_0 e^{-\delta T} N(-d_1)$
$1 \mid S > K$ (CONC)	$e^{-rT} N(d_2)$
$1 \mid S < K$ (CONP)	$e^{-rT} N(-d_2)$

Relationship to Standard Options
- $C = S_0 e^{-\delta t} N(d_1) - K e^{-rt} N(d_2) = (S \mid S > K) - (K \mid S > K)$
- $P = K e^{-rt} N(-d_2) - S_0 e^{-\delta t} N(-d_1) = (K \mid S < K) - (S \mid S < K)$
- Also note that:
  - $(S \mid S > K) = S_0 \Delta_C$
  - $(S \mid S < K) = -S_0 \Delta_P$

Gap Options

Strike and Trigger Prices
- Strike Price = K_1, Trigger Price = K_2

Payoff
- Gap Call PO = $S_T - K_1$ if $S_T > K_2$
- Gap Put PO = $K_1 - S_T$ if $S_T < K_2$
- PO could be negative. Exercise is not optional.

Pricing Gap Options
- $\text{GapCall} = S_0 e^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2)$
- $\text{GapPut} = K_1 e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$
- Use K_2 when calculating d_1 and d_2.

Parity Relation
- $\text{GapCall} - \text{GapPut} = S_0 e^{-\delta T} - K_1 e^{-rT}$

Forward Start Options

- $d_1 = \dfrac{-\ln c + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$
- $d_2 = d_1 - \sigma\sqrt{T - t}$
- $C_{FS} = S_0 e^{-\delta T} N(d_1) - c S_0 e^{-\delta t} e^{-r(T-t)} N(d_2)$
- $P_{FS} = c\, S_0 e^{-\delta t} e^{-r(T-t)} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$

Maxima and Minima

- $\max[A,\ B] = B + \max[A - B,\ 0] = A + \max[0,\ B - A]$
- $\min[A,\ B] = B + \min[A - B,\ 0] = A + \min[0,\ B - A]$
- $\max[kA,\ kB] = k\max[A,\ B]$ if $k > 0$
- $\min[kA,\ kB] = k\min[A,\ B]$ if $k > 0$
- $\max[-A,\ -B] = -\min[A,\ B]$
- $\max[A,\ B] + \min[A,\ B] = A + B$
- $\min[A,\ B] = A + B - \max[A,\ B]$

16,17 – BROWNIAN MOTION

Standard Brownian Motion

Properties

- $Z(0) = 0$
- $Z(t) \sim \text{Normal}\left(m = 0,\ v^2 = t\right)$
- $Z(t+h) \mid Z(t) \sim \text{Normal}\left(m = Z(t),\ v^2 = h\right)$
- If $[a, b]$ and $[c, d]$ don't overlap, then $Z(b) - Z(a)$ and $Z(d) - Z(c)$ are independent.

Arithmetic Brownian Motion

- ***Definition***: $A(t) = A(0) + \mu t + \sigma Z(t)$
- ***Differential***: $dA = \mu\, dt + \sigma\, dZ$

Properties

- $A(t) \sim \text{Normal}\left(m = A(0) + \mu t,\ v^2 = \sigma^2 t\right)$
- $A(t+h) \mid A(t) \sim \text{Normal}\left(m = A(t) + \mu h,\ v^2 = \sigma^2 h\right)$
- $\text{Cov}\left[A(t_1),\ A(t_2)\right] = \sigma^2 \min\left(t_1,\ t_2\right)$

Geometric Brownian Motion

- ***Definition***: $G(t) = e^{A(t)}$ where $A(t) = A(0) + \mu t + \sigma Z(t)$
- ***Differential***: $dG = \left(\mu + 0.5\sigma^2\right) G\, dt + \sigma G\, dZ$

Equivalent Expressions for GBM

- $dG = \left(\mu + 0.5\sigma^2\right) G\, dt + \sigma G\, dZ$
- $\dfrac{dG}{G} = \left(\mu + 0.5\sigma^2\right) dt + \sigma\, dZ$
- $G(t) = e^{A(0) + \mu t + \sigma Z(t)}$
- $G(t) = G(0) e^{\mu t + \sigma Z(t)}$
- $d \ln G(t) = \mu\, dt + \sigma\, dZ(t)$
- $\ln G(t) = A(0) + \mu t + \sigma Z(t)$

Stock Model

- ***Return:*** $A(t) = \left(\alpha - \delta - 0.5\sigma^2\right) t + \sigma Z(t)$
- ***Stock Price***: $S(t) = S(0) e^{A(t)}$
- ***Differential***: $dS = (\alpha - \delta) S\, dt + \sigma S\, dZ$

Equivalent Expressions for Stock Model

- $S(t) = S(0) e^{\left(\alpha - \delta - 0.5\sigma^2\right)t + \sigma Z(t)}$
- $dS(t) = (\alpha - \delta) S(t)\, dt + \sigma S(t)\, dZ(t)$
- $\dfrac{d S(t)}{S(t)} = (\alpha - \delta)\, dt + \sigma\, dZ$
- $d\left[\ln S(t)\right] = \left(\alpha - \delta - 0.5\sigma^2\right) dt + \sigma\, dZ$
- $\dfrac{d F^P(S)}{F^P(S)} = (\alpha - \delta)\, dt + \sigma\, dZ$
- $d\left[\ln F^P(S)\right] = \left(\alpha - \delta - 0.5\sigma^2\right) dt + \sigma\, dZ$
- $\ln\left[\dfrac{S(t)}{S(0)}\right] \sim \text{Normal}\left[m = \left(\alpha - \delta - 0.5\sigma^2\right) t,\ v^2 = \sigma^2 t\right]$

18 – ITO'S LEMMA

Ito's Lemma

- ***Multiplication Rules***: $(dt)^2 = dt\, dZ = 0$, $(dZ)^2 = dt$
- ***Ito's Lemma***: $dV = V_t\, dt + V_s\, dS + 0.5\, V_{SS} (dS)^2$

19 – BLACK-SCHOLES EQUATION

Black-Scholes Equation

- $(r - \delta) S V_S + 0.5\sigma^2 S^2 V_{SS} + V_t = \left(r - \delta^*\right) V$
- δ^* is the rate of dividends paid by the derivative itself.

20 – SHARPE RATIO

Sharpe Ratio

- $\phi = \dfrac{\alpha - r}{\sigma}$
- If *A* and *B* are assets driven by the same dZ, then $|\phi_A| = |\phi_B|$.

Risk-Free Portfolios

- Let $\dfrac{d X_1}{X_1} = (\alpha_1 - \delta_1) dt + \sigma_1\, dZ$ and $\dfrac{d X_2}{X_2} = (\alpha_2 - \delta_2) dt + \sigma_2\, dZ$.
- Purchase c_1 shares of X_1 and c_2 shares of X_2, where:
- $c_1 X_1(0)\sigma_1 + c_2 X_2(0)\sigma_2 = 0$
- $\dfrac{c_1 X_1(0)\alpha_1 + c_2 X_2(0)\alpha_2}{c_1 X_1(0) + c_2 X_2(0)} = r$

21 – RISK-NEUTRAL PRICING AND PROP. PORTFOLIOS

Risk-Neutral Pricing

- True process: $dS = (\alpha - \delta) S\, dt + \sigma S\, dZ$
- R-N process: $dS = (r - \delta) S\, dt + \sigma S\, d\tilde{Z}$
- $d\tilde{Z} = dZ + \phi\, dt$ and $\tilde{Z}(t) = Z(t) + \phi t$, where $\phi = \dfrac{\alpha - r}{\sigma}$

Expected Values

- True: $E\left[Z(t)\right] = 0$, $E\left[\tilde{Z}(t)\right] = \phi t$, and $E\left[S(t)\right] = S(0) e^{(\alpha - \delta)t}$
- R-N: $E^*\left[\tilde{Z}(t)\right] = 0$, $E^*\left[Z(t)\right] = -\phi t$, and $E^*\left[S(t)\right] = S(0) e^{(r - \delta)t}$

Proportional Portfolios

Let $W(t)$ be the value of a portfolio that always has $100p\%$ of its value invested in a stock following $dS = \left(\alpha - \delta_S\right) S\, dt + \sigma S\, dZ$ and $100(1-p)\%$ of its value invested in a bond following $dB = r B\, dt$.

- $\dfrac{dW}{W} = \left[p\alpha + (1 - p) r - \delta_W\right] dt + p\sigma\, dZ$
- $W(t) = W(0) e^{\left[p\alpha + (1-p) r - \delta_W - 0.5 p^2 \sigma^2\right] t + p\sigma Z(t)}$
- $W(t) = W(0) \left[\dfrac{S(t)}{S(0)}\right]^p e^{\left[p\,\delta_S - \delta_W + (1 - p)\left(r + 0.5\, p\sigma^2\right)\right] t}$

22 – POWERS OF S

Expected value of $S(T)^a$

- True Probability: $E\left[S(T)^a\right] = S(0)^a e^{\left[a\left(\alpha - \delta - 0.5\sigma^2\right) + 0.5 a^2 \sigma^2\right] T}$
- R-N Probability: $E^*\left[S(T)^a\right] = S(0)^a e^{\left[a\left(r - \delta - 0.5\sigma^2\right) + 0.5 a^2 \sigma^2\right] T}$

Forwards on S^a

- $F_{0,T}\left(S^a\right) = S(0)^a e^{\left[a(r - \delta) + 0.5 a(a - 1)\sigma^2\right] T}$
- $F_{0,T}^P\left(S^a\right) = e^{-rT} S(0)^a e^{\left[a(r - \delta) + 0.5 a(a - 1)\sigma^2\right] T}$

Ito Process for S^a

- $\dfrac{d S(t)^a}{S(t)^a} = \left[a(\alpha - \delta) + 0.5\, a(a - 1)\sigma^2\right] dt + a\sigma\, dZ$
- $S(t)^a = S(0)^a e^{B(t)}$, where $B(t) = a\left(\alpha - \delta - 0.5\sigma^2\right) t + a\sigma Z(t)$

Dividend Yield for S^a

- If $V(t) = S(t)^a$, then $\delta^* = r - a(r - \delta) - 0.5 a(a - 1)\sigma^2$.

24 – TREE MODELS FOR INT. RATES

Notation for Bond Prices

- $P(t,T)$ = time t price of bond worth \$1 at time T.
- $F_{0,t}(t,T) = P_0(t,T)$ = forward price of $P(t,T)$.
- $F_{0,t}(t,T) = P_0(t,T) = \dfrac{P(0,T)}{P(0,t)}$

Black-Derman-Toy Model

- Rates are effective, and $p^* = 0.5$.
- Let $r_{t,k}$ be the rate for the time t node, k nodes above the bottom.
- $r_{t,k+1} = r_{t,k}e^{2\sigma_t\sqrt{h}}$, where σ_t is the short term volatility.

Long-Term Volatility

- $y_u(1,T) = P_u(1,T)^{-(T-1)} - 1$ and $y_d(1,T) = P_d(1,T)^{-(T-1)} - 1$
- Volatility in the price of $P(1,T)$ is $\sigma_{1,T} = \dfrac{1}{(T-1)\sqrt{h}}\ln\left[\dfrac{y_u(1,T)}{y_d(1,T)}\right]$

Pricing Caps

- L is the loan amount, and r_k is the cap.
- Caplet value = $\max[0, L(r_{t,i} - r_k)]$.
- Cap value is probability weighted sum of the PV of all caplets.

26 – CONTINUOUS INT. RATE MODELS

General Equilibrium Model

- ***Rate Process:*** $dr(t) = P(r)dt + Q(r)dZ(t)$
- ***Bond Process:*** $\dfrac{dP(r,t,T)}{P(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ(t)$
- ***Sharpe Ratio:*** $\phi(r,t,T) = \dfrac{\alpha(r,t,T) - r}{q(r,t,T)}$

Rendleman-Bartter Model

- ***Rate Process:*** $dr = ar\,dt + \sigma r\,dZ$
  - No mean reversion, $r \ge 0$, σ is proportional to r.
  - r follows geometric Brownian Motion

Vasicek Model

- ***Rate Process:*** $dr = a(b-r)dt + \sigma dZ$
  - Mean reversion, r can become negative, σ is constant.
- ***Bond Prices:*** $P(0,T) = A(0,T)e^{-B(0,T)r(t)}$
  - $A(0,T) = ???$, $B(0,T) = \dfrac{1}{a}\left(1 - e^{-aT}\right)$
  - $A(h,T+h) = A(0,T)$ and $B(h,T+h) = B(0,T)$
- ***Sharpe Ratio:***
  - ϕ is constant in this model
  - $q(r,t,T) = B(t,T)\sigma$

Cox-Ingersoll-Ross Model

- ***Rate Process:*** $dr = a(b-r)dt + \sigma\sqrt{r}\,dZ$
  - Mean reversion, $r \ge 0$, σ is proportional to $\sqrt{r}$.
- ***Bond Prices:*** $P(0,T) = A(0,T)e^{-B(0,T)r(t)}$
  - $A(0,T) = ???$, $B(0,T) = ???$
  - $A(h,T+h) = A(0,T)$ and $B(h,T+h) = B(0,T)$
- ***Sharpe Ratio:***
  - $\phi(r,t,T) = \dfrac{\bar{\phi}}{\sigma}\sqrt{r}$, where $\bar{\phi}$ is a constant.
  - $q(r,t,T) = B(t,T)\sigma\sqrt{r}$

26 – HEDGING FORMULAS

Bond Hedging Formulas

Bond 1 expires at time T_1 and Bond 2 expires at time T_2. You buy 1 unit of Bond 1 at time t and hedge by buyinh N units of Bond 2.

- **Duration-Hedge:** $N = -\dfrac{(T_1 - t)P(t,T_1)}{(T_2 - t)P(t,T_2)}$
- **Delta-Hedge:** $N = -\dfrac{P_r(r,t,T_1)}{P_r(r,t,T_2)} = -\dfrac{B(t,T_1)P(r,t,T_1)}{B(t,T_2)P(r,t,T_2)}$

15 – MONTE CARLO VALUATION

Simulating Derivative Prices

Standard Normal Random Numbers

- $z_i = \left(\sum_{i=1}^{12} u_j\right) - 6$, $U \sim \text{Uniform}(0,1)$
- $z_i = N^{-1}(u_i)$, $U \sim \text{Uniform}(0,1)$

Lognormal Random Numbers

- $n_i = m + z_i v$, $N \sim \text{Normal}(m, v^2)$
- $x_i = e^{n_i}$, $X = e^N \sim \text{LogN}(m, v^2)$

Simulating Stock Prices

- We use risk-neutral pricing.
- $m = (r - \delta - 0.5\sigma^2)T$ and $v = \sigma\sqrt{T}$
- $S_T^i = S_0 e^{m + z_i v}$

Simulating Option Price

- For each S_T^i, find the option payoff V_T^i.
- $V_i = e^{-rT}V_T^i$
- $\bar{V} = \sum V_i$

Control Variate Methods

- V = option being priced
- K = option with known price, K_0

Basic Control Variate Method

- $V^* = \bar{V} + K_0 - \bar{K}$
- $\text{Var}[V^*] = \text{Var}[\bar{V}] + \text{Var}[\bar{K}] - 2\,\text{Cov}[\bar{V}, \bar{K}]$

Boyle Modification

- $\beta = \dfrac{\text{Cov}[V, K]}{\text{Var}[K]} = \dfrac{\left(\sum v_i k_i\right) - n\bar{V}\bar{K}}{\left(\sum k_i^2\right) - n\bar{K}^2}$
- $V^{BM} = \bar{V} + \beta(K - \bar{K})$
- $\text{Var}[V^{BM}] = \text{Var}[\bar{V}] + \beta^2\text{Var}[\bar{K}] - 2\beta\,\text{Cov}[\bar{V}, \bar{K}]$
  $= \text{Var}[\bar{V}]\left(1 - \rho_{\bar{V}\bar{K}}^2\right)$
- Recall that $\rho_{\bar{V}\bar{K}} = \dfrac{\text{Cov}[\bar{V}, \bar{K}]}{\sqrt{\text{Var}[\bar{V}]\text{Var}[\bar{K}]}}$.
- Tip: Use LinReg in TI-30XS to find β.

Antithetic and Stratified Sampling

- Assume $u_1, u_2, \ldots, u_n$ have been generated.

Antithetic Method

- For each i, add $u_i^* = 1 - u_i$ to the sample.
- For each i, add $n_i^* = -n_i$ to the sample.

Stratified Sampling

- For each i, add $u_i^* = \dfrac{i-1}{n} + \dfrac{u_i}{n}$ to the sample.